MATHEMATICAL PROBLEMS IN ESTIMATION THEORY & CONFIDENCE INTERVAL

K.M. MANIKANDAN

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MATHEMATICAL PROBLEMS IN ESTIMATION THEORY & CONFIDENCE INTERVAL

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CHAPTER I
STATISTICAL INFERENCE
(THEORY OF ESTIMATION)

	Page No
1.1 Concept Of Population	1
1.2 Estimation	2
1.3 Consistency	5
1.4. Unbiasedness	10
1.5 Sufficiency	23
1.6 Efficient Estimators	27
1.7 Factorization Theorem Due To Neyman.	29
1.8 Cramer Rao Inequality	30
1.9 Rao - Blackwell Theorem	36
1.10Exercise	41

CHAPTER II
METHODS OF ESTIMATION AND CONFIDENCE INTERVAL

2.1 Methods Of Maximum Likelihood Estimation	44
2.2 Method Of Moments	80
2.3 Method Of Least Squares	94
2.4 Method Of Minimum Variance	95
2.5 Method Of Minimum Chi – Square Estimation	96
2.6 Confidence Interval Of The Mean	98
2.7 Confidence Interval Estimate Of The Proportion	103
2.8 Confidence Intervals	109
2.9 Confidence Interval For Population Mean For Small Samples Using 't' Distribution	134
2.10Confidence Intervals For Large Samples	139
2.11 Exercise	145

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DEDICATED THIS BOOK TO MY PARENTS

Mr. K.N. Murukaiyan

&

Mrs. K.M. Vijayalakshmi

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PREFACE

This book has been written for the benefit of the students society to learn the basic concepts and the related problems in the areas of Estimation theory and confidence intervals. Adequate number of examples with detailed explanation has been given in this book.

I sincerely dedicate this book to my parents. I thank our college management, **Dr.S.N. Subramanian,** Chairman, **Dr.V.S. Velusamy,** Secretary, **Dr. S. Rajalakshmi,** Correspondent for their continuous support during this work.

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Suggestions for the improvement of this book are welcome

K.M. MANIKANDAN

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CHAPTER I

STATISTICAL INFERENCE (THEORY OF ESTIMATION)

1.1 : CONCEPT OF POPULATION

In day today life investigations are being carried out on several fields such as sales on daily consumer products, demand of durable goods census calculation, animals and plants life on the earth, ratings of serials on different TV channels etc. Based on one or more characteristics datas are collected from various sources through investigators working in government bodies and private media. Then the collected datas are devised, scrutinized, and the results are inferred. The said, various sources completely (or) partially considered for this study are collectively known as **universe (or) population.**

Unit :

A smallest non-divisible part of the population is called a **unit**. A unit should be well defined and should be unambiguous. For example, if we define unit as a household, then it should be defined that a person should not belong to two house holds.

Sample :

A finite subset of a population is called a **sample** and the number of units in a sample is called its **sample**

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size.

By analyzing the data collected from the sample, one can, draw inference about the population under study.

Parameter and statistic

The statistical constants of a population like mean (μ), variance (σ^2) proportion (P) are termed as **parameters.** Statistical measures like mean ($\bar{x}$), variance (s^2) proportion (p) computed from the sample observations are known as **statistic.**

1.2 ESTIMATION

Introduction

The technique used for generalizing the result of sample to the population is provided by an important branch of statistics called **statistical inference**. The concept of statistical inference deals with two basic aspects namely (a) **Estimation** and (b) **testing of hypothesis**. In statistics, estimation is concerned with making inference about the parameters of the population using available information. This chapter provides some insights into point estimation and internal estimation.

Estimator

As **estimator** is a sample statistic used to estimate a population parameter. For example, sample

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mean ($\bar{x}$) is the estimator of the population Mean (μ). Here we use $\bar{x}$ to estimate the population parameter μ. We call the numerical value of $\bar{x}$ as a **point estimate** of the parameter μ.

Examples:

i. An observed value of $\bar{x}$ to estimate the mean of population is a point estimate and the statistic $\bar{x}$ is a point estimator.

ii. A value of s^2 to estimate a population variance is called a point estimate and the statistic s^2 is called an estimator.

iii. An observed sample proportion to estimate the parameter of the binomial distribution is a point estimator and the statistic p is a point estimation.

In all point estimates we consider single number or a single point on the real axis to estimate the parameter.

Estimators are random variables which are used to obtain the sampling distribution of statistics.

Point Estimate

An estimate of a population parameter given by a single number is called a point estimator of the parameter. Mean

Mean ($\bar{x}$) and the sample variance $s^2 = \frac{1}{n-1}\sum(x-\bar{x})^2$ are the example of point estimators

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A point estimate will rarely coincide with true population parameter value

Interval estimate:

An estimate of a population parameter given by two numbers between which the parameter is expected to lie is called an **interval estimate** of the parameter.

Interval estimate indicates the accuracy of an estimate and is there fore preferable. to point estimate. A point estimate provides a single value for the population parameter it may not be suitable in some situation.

For example, If we say that a distance is measured as 5.28 mm, we are given point estimate. On the other hand if we say that the distance is 5.28 ± 0.03 mm (ie) the distance lies between 5.25 and 5.31 mm we are given an interval estimate.

CHARACTERISTICS OF ESTIMATORS:

A good estimator is one which is as close to the true value of a population parameter as possible. A good estimator possesses the following properties.

i. Consistency

ii. Unbiasedness

iii. Sufficiency

iv Efficiency

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Example : 1.2.1

A sample of 5 measurements of the diameters of a sphere were recorded by a scientist as 6.33, 6.37, 6.36, 6.32 and 6.37 mm. Determine the point estimate of (a) mean (b) variance

Solution

x	$(x-\bar{x})$	$(x-\bar{x})^2$
6.33	-0.02	0.0004
6.37	0.02	0.0004
6.36	0.01	0.0001
6.32	-0.03	0.0009
6.37	0.02	0.0004
31.75		0.0022

Sample mean $\bar{x} = \frac{\Sigma x}{n} = \frac{31.75}{5} = 6.35$

Sample variance $s^2 = \frac{1}{n-1} \Sigma (x-\bar{x})^2$

$$= \frac{0.0022}{5-1} = 0.0055$$

1.3CONSISTENCY

An estimator $T_n = T(x_1,x_2.....x_n)$ based on a random sample of size 'n', is said to be **consistent estimator** of $\gamma(\theta)$, $\theta \in \Theta$, the parameter space, if T_n converges to $\gamma(\theta)$ in

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probability (i.e) if $T_n \xrightarrow{P} \gamma(\theta)$ as $n \to \infty$. ...(1)

In other words, T_n is a consistent estimator of $\gamma(\theta)$ if for every $\epsilon > 0$, $\eta > 0$, there exists a positive integer $n \geq m$ (ϵ, η) such that

$$P[\,|T_n - \gamma(\theta)| < \epsilon] \to 1 \text{ as } n \to \infty. \qquad ....(2)$$

$$\Rightarrow P[\,|T_n - \gamma(\theta)| < \epsilon] > 1 - \eta, \ \forall\, n \geq m \qquad ....(3)$$

where m is very large value of n.

Note :

Let us take a random sample $x_1, x_2 .....x_n$ of size 'n' from a population with finite mean $E(x_i) = \mu < \infty$, then by Khinchine's weak law of large numbers (W.L.L.N)

we have

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} x_i \xrightarrow{P} E(x_i) = \mu$$

Hence sample mean $(\overline{X}_n)$ is always a consistent estimator of the population mean (μ).

INVARIANCE PROPERTY OF CONSISTENT ESTIMATORS

Theorem 1.3.(i)

Statement :

If T_n is a consistent estimator of $\gamma(\theta)$ and $\psi(\gamma(\theta)$ is a continuous function of $\gamma(\theta)$, then. $\psi(T_n)$ is consistent estimator of $\psi(\gamma(\theta)$.

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Proof :

Since T_n is a consistent estimator if $\gamma(\theta)$, we have

$T_n \xrightarrow{p} \gamma(\theta)$ as $n \to \infty$

(i.e) for every $\varepsilon > 0, \eta > 0, \exists$ +ve integer $n \geq m(\varepsilon,\eta)$ such that

$P[|T_n - \gamma(\theta)| < \varepsilon] > 1 - \eta, \forall\ n \geq m$ .....(1)

Since $\psi(\gamma(\theta)$ is a continuous function of $\gamma(\theta)$, for every

$\varepsilon > 0$, however small $\exists$ + ve integer ε_1 such that

$|\psi(T_n) - \psi(\gamma(\theta))| < \varepsilon_1$ whenever $|T_n - \gamma(\theta)| < \varepsilon$

(i.e) $|T_n - \gamma(\theta)| < \varepsilon \Rightarrow |\psi(T_n) - \psi(\gamma(\theta)| < \varepsilon_1$ .... (2)

For two events A and B

If $A \Rightarrow B$, then

$A \subseteq B \Rightarrow P(A)\ ?\ P(B) \Rightarrow P(B) \geq P(A)$ ..... (3)

From (2) and (3)

$P[|\psi(T_n) - \psi(\gamma(\theta))| < \varepsilon_1] \geq P[|T_n - \gamma(\theta)| < \varepsilon]$

$\Rightarrow P[|\psi(t_n) - \psi(\gamma(\theta))| < \varepsilon_1] \geq 1 - \eta \forall\ n \geq m$ .....using (1)

$\Rightarrow \psi(T_n) \xrightarrow{p} \psi(\gamma(\theta))$ as $n \to \infty$

$\Rightarrow \psi(T_n)$ is a consistent estimator of $\gamma(\theta)$

Sufficient Conditions For Consistency

Theorem : 1.3.(ii)

Let $\{T_n\}$ be a sequence of estimators such that for all $\theta \in$?

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(i) $E_\theta(T_n) \to \gamma(\theta)$, $n \to \infty$

and (ii) $Var_\theta(T_n) \to 0$, as $n \to \infty$

Then T_n is consistent estimator of $\gamma(\theta)$.

Proof :

We have to P.T. T_n is a consistent estimator of $\gamma(\theta)$

(i.e) $T_n \xrightarrow{p} \gamma(\theta)$ as $n \to \infty$

$P[\,|T_n - \gamma(\theta)| < \varepsilon] > 1 - \eta \quad \forall\, n \geq m(\varepsilon, \eta)$ ....(1)

Where ε and η are arbitrary small +ve numbers and m is some large value of n.

Applying Chebyshev in equality to the statistic T_n we have

$$P[|T_n - E_\theta(T_n)| \leq \delta] \geq 1 - \frac{Var_\theta(T_n)}{\delta^2} \qquad \ldots (2)$$

We have

$$|T_n - \gamma(\theta)| = |T_n - E(T_n) + E(T_n) - \gamma(\theta)|$$

$$\leq |T_n = E_\theta(T_n) + E_\theta(T_n) - \gamma(\theta)| \qquad \ldots.(3)$$

Now $|T_n - E_\theta(T_n)| \leq \delta$

$$\Rightarrow |T_n - \gamma(\theta)| \leq \delta + |E_\theta(T_n) - \gamma(\theta)| \qquad \ldots(4)$$

Using the idea

If $A \Rightarrow B$ then $A \subseteq B \Rightarrow P(B) \geq P(A)$

$P[|T_n - \gamma(\theta)| \leq \delta + E_\theta(T_n) - \gamma(\theta)]$

$$\geq P[T_n - E_\theta(T_n) \leq \delta] \geq 1 - \frac{Var_\theta(T_n)}{\delta^2} \quad \to (5)$$

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We are given

$E_\theta(T_n) \to \gamma(\theta)\ \forall\ \theta \in \Theta$ as $n \to \infty$

Hence $\forall\ \delta_1 > 0$, $\exists$ positive integer $n \geq n_o(\delta_1)$ such that

$$|E_\theta(T_n) - \gamma(\theta)| \leq \delta_1\ \ \forall\ n \geq n_o(\delta_1) \qquad \to (6)$$

Also

$Var_\theta(T_n) \to 0$ as $n \to \infty$ (given)

$$\therefore\ \frac{Var_\theta(T_n)}{\delta^2} \leq \eta,\ \forall n \geq n_{o'}(\eta) \qquad \to (7)$$

where η is arbitrarily small positive number substituting values of equations (6) and (7) in (5)

$P[|T_n-\gamma(\theta)| \leq \delta+\delta_1] \geq 1-\eta \quad n \geq m(\delta_1, \eta)$

$\Rightarrow P[|(T_n)-\gamma(\theta)| \leq \varepsilon] \geq 1-\eta \quad n \geq m$

where $m = \max(n_o, n_{o'})$ and $\varepsilon = \delta + \delta_1 > 0$

$\Rightarrow T_n \xrightarrow{p} \gamma(\theta)$ as $n \to \infty$

$\Rightarrow T_n$ is a consistent estimator of $\gamma(\theta)$

Example : 1.3.1

Prove that in sampling from $N(\mu,\sigma^2)$ population, the sample mean is a consistent estimator of μ.

Solution :

In sampling from normal population with mean μ and variance σ^2, we have $E(\bar{x}) = \mu$ and $var(\bar{x}) = \frac{\sigma^2}{n}$

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When n $\rightarrow \infty$

$E(\bar{x}) = \mu$ and var $(\bar{x}) = 0$.

Hence. by theorem (1.3.(ii)), $\bar{x}$ is a consistent estimator for μ.

1.4. UNBIASEDNESS

A statistic $\hat{\theta}$ is said to be an ***unbiased*** estimator of parameter. θ if and only if $E(\hat{\theta}) = \theta$. $\hat{\theta}$ is said to be **biased** if $E(\hat{\theta}) \neq \theta$. The quantity $E(\hat{\theta}) - \theta$ is called the **bias** of the estimator. $\hat{\theta}$ is said to be **positively biased** if $E(\hat{\theta}) - \theta > 0$ and if $E(\hat{\theta}) - \theta < 0$ then $\hat{\theta}$ is said to be negatively biased. $\hat{\theta}$ is said to be asymptotically unbiased estimator of θ if $\underset{n\to\infty}{Lt}\ E(\hat{\theta}) = \theta$

An unbiased estimator may be meaningless, do not always exist and may not be unique.

Example 1.4.1

Let $x_1, x_2 \ldots\ldots\ldots x_n$ be a simple random sample from a population with man μ . Then show that sample mean is an unbiased estimator of the population mean

Solution

Let $\bar{x} = \dfrac{x_1 + x_2 \ldots\ldots + x_n}{n}$ and μ be a Population mean

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Since $x_1, x_2 \ldots x_n$ is a simple random sample with mean μ

$E(x_1) = E(x_2) \ldots\ldots\ldots\ldots\ldots\ldots = E(x_n) = \mu$

Claim : $\bar{x}$ is an unbiased estimator of μ, we have to prove that $E(\bar{x}) = \mu$. Here

$$E(\bar{x}) = E\left(\frac{x_1 + x_2 + \ldots\ldots + x_n}{n}\right)$$

$$= \frac{1}{n} E(x_1 + x_2 + \ldots\ldots + x_n)$$

$$= \frac{1}{n} [E(x_1) + E(x_2) + \ldots\ldots + E(x_n)]$$

$$= \frac{1}{n} (n\mu)$$

$$= \mu$$

Hence $\bar{x}$ is an unbiased estimator of μ.

Example 1.4.2

Show that the sample variance is not an unbiased estimator of the population variance but an asymptotically unbiased estimator of σ^2.

Let $s^2 = \dfrac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}$ be the sample variance

$$= \frac{1}{n}\left[\sum_{i=1}^{n}(x_i^2 - 2x_i\bar{x} + \bar{x}^2)\right]$$

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$$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - 2\bar{x}\frac{1}{n}\sum x_i + \frac{1}{n}\sum \bar{x}^2$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - 2\bar{x}.\bar{x} + \frac{1}{n}.n\bar{x}^2$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - 2\bar{x}^2 + \bar{x}^2$$

$$s^2 \quad = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2$$

$$E(s^2) \quad = \frac{1}{n}E\left[\sum_{i=1}^{n} x_i^2 - \bar{x}^2\right]$$

$$= \frac{1}{n}E\left[\sum_{i=1}^{n} x_i^2 - E(\bar{x}^2)\right]$$

$$= \frac{1}{n}E\left[\sum_{i=1}^{n} n\bar{x}^2\right] - E(\bar{x})^2$$

$$= \frac{1}{n}E\left[nx^2\right] - E(\bar{x})^2$$

$$= \frac{1}{n}n\ E(x^2) - E\ (\bar{x})^2$$

$$E(s^2) \quad = E\ (x^2) - E(\bar{x}^2)\ \ldots\ldots\ldots\ldots(1)$$

$$\text{Now } E\ (\bar{x}^2) \quad = E\left[\left(\frac{x_1 + x_2 + \ldots\ldots\ldots\ldots + x_n}{n}\right)^2\right]$$

$$= \frac{1}{n^2}E[\sum x_i^2 + \sum x_i x_j]$$

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$$= \frac{1}{n^2}\left[E\left(\sum x_i^2\right) + E\left(\sum x_i x_j\right)\right]$$

$$= \frac{1}{n^2}\left[n.E\left(X^2\right) + \left(n(n-1)E(\bar{x}\right)^2\right]$$

$$= \frac{1}{n}.E\left(x^2\right) + \frac{n-1}{n}[E\{\bar{x})]^2 \qquad \text{........ (2)}$$

Sub (2) in (1)

$$E(\bar{x}^2) = E(x^2) - \left[\frac{1}{n}E(x^2) + \frac{(n-1)}{n}E(\bar{x})^2\right]$$

$$= E(x^2) - \frac{1}{n}E(x^2) - \frac{(n-1)}{n}[E(\bar{x})]^2$$

$$= \frac{(n-1)}{n}E(x^2) - \frac{(n-1)}{n}\left[E(\bar{x})\right]$$

$$= (n)\left[E(x^2) - (E(\bar{x})^2\right]$$

$$E(s^2) = \left(\frac{n-1}{n}\right)\sigma^2$$

Since $E(s^2) \neq \sigma^2$, Sample variance is not an unbiased estimate of the population variance.

$$\text{But } \lim_{n\to\infty} E(s^2) = \lim_{n\to\infty}\left(\frac{n-1}{n}\right)\sigma^2$$

$$= \lim_{n\to\infty}\left(1-\frac{1}{n}\right)\sigma^2$$

$$= \sigma^2$$

Therefore s^2 is an asymptotically unbiased estimator of σ^2.

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Example 1.4.3

Let $x_1, x_2 \ldots\ldots x_n$ is an unbiased sample from the population with respect to density function

$f(x) = \exp[-(x-\theta)]$ for $x > \theta$

$= 0$ otherwise

Show that $\bar{x}$ is an unbiased estimator of θ

Solution :

Given $f(x) = \exp[-(x-\theta)]$ for $x > \theta$

$= 0$, otherwise

$$E(\bar{x}) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{\theta}^{\infty} x \exp[-(x-\theta)]\, dx$$

$$= \left[x \frac{\exp[-(x-\theta)]}{-1} \right]_{\theta}^{\infty} - \int_{\theta}^{\infty} \exp[-(x-\theta)]\, dx$$

$$= \theta + 1$$

Since $\bar{x} = E(x) \neq \theta \therefore \bar{x}$ is not an unbiased estimator of θ

But $E(\bar{x}-1) = E(\bar{x}) - E(1)$

$= \theta + 1 - 1$

$= \theta$

$\therefore \bar{x}-1$ is an unbiased estimator of θ

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Example 1.4.4

A random sample (x_1, x_2, x_3, x_4) of size 4 drawn from a Poisson population with a parameter m. S.T. (i) $\frac{x_1 + x_2 + x_3 + x_4}{4}$ is an unbiased estimator of λ. (ii) Find ν such that $\frac{\nu x_1 + x_2}{2}$ is an unbiased estimator for λ.

Solution :

For a poisson distribution $\mu = \lambda$

$E(x_i) = \lambda \quad i = 1,2,3,4$

i. $$E\left(\frac{x_1 + x_2 + x_3 + x_4}{4}\right) = \frac{1}{4}\left[E(x_1)+E(x_2)+E(x_3)+E(x_4)\right]$$

$$= \frac{1}{4}(4\lambda) = \lambda$$

$\Rightarrow \frac{x_1 + x_2 + x_3 + x_4}{4}$ is an unbiased estimator of λ.

ii. $$E\left(\frac{\nu x_1 + x_2}{2}\right) = \lambda$$

$$\frac{1}{2}\left[\nu E(x_1) + E(x_2)\right] = \lambda$$

$$\left[\nu E(x_1) + E(x_2)\right] = 2\lambda$$

$$\nu\lambda + \lambda = 2\lambda$$

$$\nu\lambda = \lambda, \ \nu = 1$$

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Example 1.4.5

If $x_1, x_2 \ldots\ldots x_n$ is a random sample from a normal population $N(\mu,1)$, show that $t = \frac{1}{n}\sum_{i=1}^{n} x_i^2$ is an unbiased estimator of $\mu^2 + 1$.

Solution :

We are given

$E(x_1) = E(x_2) \ldots\ldots\ldots\ldots\ldots\ldots = E(x_n) = \mu.$

and $\text{var}(x_1) = \text{var}(x_2) \ldots\ldots\ldots\ldots = \text{Var}(x_n) = 1$

now $E(x_1^2) = \text{var}(x_1) + [E(x_1)]^2 = 1 + \mu^2$

$E(x_2^2) = \ldots\ldots E(x_n^2) = 1 + \mu^2$

$$E(t) = E\left[\frac{1}{n}\sum_{i=1}^{n} x_i^2\right]$$

$$= \frac{1}{n}\left[\sum_{i=1}^{n} E(x_i^2)\right]$$

$$= \frac{1}{n}\left[(1+\mu^2)\ldots\ldots\ldots n \text{ times}\right]$$

$$= 1 + \mu^2$$

Therefore $t = \frac{1}{n}\sum_{i=1}^{n} x_i^2$ is an unbiased estimator of μ^2+1.

Example : 1.4.6

If T is an unbiased estimator for θ show that. T^2 is a biased estimator for θ^2.

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Solution :

Since T is an unbiased estimator for θ, we have $E(T) = \theta$.

$$\text{var}(T) = E(T^2) - [E(T)]^2$$
$$= E(T^2) - \theta^2.$$
$$\Rightarrow E(T^2) = \text{var}(T) + \theta^2$$

Since $E(T^2) \neq \theta^2$, T^2 is not an unbiased estimator for θ^2.

Example 1.4.7

Show that $\frac{\sum x_i(\sum x_i - 1)}{n(n-1)}$ is an unbiased estimate of p^2, for the sample $x_1, x_2 \ldots\ldots\ldots\ldots\ldots x_n$ drawn on X which takes the values 1 or 0 with respective probabilities p and 1-p.

Solution :

Since $x_1, x_2 \ldots\ldots\ldots x_n$ is a random sample. From Bernoulli population with parameter p.

$$T = \sum_{i=1}^{n} x_i \sim B(n, p)$$

$\Rightarrow E(T) = np$ and $\text{var}(T) = npq$ Let $T = \sum x_i$, then

$$E\left[\frac{\sum x_i(\sum x_i - 1)}{n(n-1)}\right] = E\left[\frac{T(T-1)}{n(n-1)}\right]$$

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$$= \frac{1}{n(n-1)}[E(T^2) - E(T)]$$

$$= \frac{1}{n(n-1)}[\text{var}(T) + [E(T)]^2 - E(T)]$$

$$= \frac{1}{n(n-1)}[npq + (np)^2 - np]$$

$$= \frac{np}{n(n-1)}[q + np - 1]$$

$$= \frac{np}{n(n-1)}[1 - p + np - 1]$$

$$= \frac{p^2(n-1)}{(n-1)} = p^2$$

$\therefore [\sum x_i(\sum x_i - 1)]/n(n-1)$ is an unbiased estimator for p^2.

Example 1.4.8

Let X be distributed in the Poisson form with parameter λ. Show that the only unbiased estimator of $e^{-(k+1)\lambda}$, $k > 0$ is $T(x) = (-k)^x$ so that $T(x) > o$ if x is even and $T(x) < 0$ if x is odd.

Solution : $E(T(x)) \quad = E[(-k)^x], \quad k > 0$

$$= \sum_{x=0}^{\infty}(-k)^x \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda}\sum_{x=0}^{\infty} \frac{(-k\lambda)^x}{x!} = e^{-(1+k)\lambda}$$

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$\Rightarrow T(x) = (-k)^x$ is an unbiased estimator for $e^{-(1+k)\lambda}$, $\lambda>0$.

Example 1.4.9

Show that $\frac{X+1}{n+2}$ is a biased estimate of the binomial parameter θ. Is this estimator asymptotically unbiased?

Solution :

We are given, for Binomial distribution, $E(x) = n\theta$.

$$E\left(\frac{X+1}{n+2}\right) = \frac{1}{n+2} E(X+1)$$

$$= \frac{E(x)+1}{n+2} [\because E(1)=1]$$

$$= \frac{n\theta+1}{n+2}$$

Since $E\left(\frac{X+1}{n+2}\right) \neq \theta$, $\frac{X+1}{n+2}$ is a biased estimator of θ.

Further

$$\underset{n\to\infty}{lt} E\left(\frac{X+1}{n+2}\right) = \underset{n\to\infty}{lt} \frac{n(\theta+1/n)}{n(1+2/n)}$$

$$= \underset{n\to\infty}{lt} \frac{\theta+1/n}{1+2/n} = \theta$$

$\Rightarrow \frac{X+1}{n+2}$ is asymtotically unbiased estimator of θ.

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Example 1.4.10

If $\bar{\theta}_1$ and $\bar{\theta}_2$ are unbiased estimators of the same parameter θ, what condition must be imposed on the constants k_1 and k_2 so that $k_1\bar{\theta}_1$ and $k_2\bar{\theta}_2$ is also an unbiased estimator θ?

Solution :

We are given, $\bar{\theta}_1$ and $\bar{\theta}_2$ are unbiased estimators of the parameter θ.

$\therefore E(\bar{\theta}_1) = \theta$ and $E(\bar{\theta}_2) = \theta$

Now,

$$E(k_1\bar{\theta}_1 + k_2\bar{\theta}_2) = k_1 E(\bar{\theta}_1) + K_2(\bar{\theta}_2)$$

$$= K_1\theta + K_2\theta$$

when $k_1 = k_2 = \dfrac{1}{2}$

$$E(k_1\bar{\theta}_1 + k_2\bar{\theta}_2) = \frac{\theta + \theta}{2} = \theta$$

$\therefore k_1\bar{\theta}_1 + k_2\bar{\theta}_2$ is also an unbiased estimator of θ when the values of k_1 and k_2 are $\dfrac{1}{2}$ each.

Example 1.4.11

Suppose that X and Y are independent random variables with unknown mean μ. Both X and Y have

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variances as 36. Let $T = aX + bY$ be an estimator of μ.

i. Show that T is an unbiased estimator of μ if $a+b=1$

ii If $a = \frac{1}{3}$ and $b = \frac{1}{2}$, what is the variance of T?

iii If $a = \frac{1}{2}$ and $b = \frac{1}{2}$, what is the variance of T?

iv What choices of a and b minimizes the variance of T subject to the requirement that T is an unbiased estimator of μ?

Solution :

we are given

$E(X) = E(Y) = \mu$

$\text{var}(X) = \text{var}(Y) = 36$ and $T = aX + bY$

(i) $$E(T) = E(aX + bY)$$
$$= aE(X) + bE(Y)$$
$$= a\mu + b\mu$$
$$= (a+b)\mu$$

When $a+b = 1$,

$$E(T) = \mu$$

(i.e) T is an unbiased estimate of μ when $a+b = 1$.

(ii) $\text{var}(T) = \text{var}(aX + bY)$

$$= \text{var}\left(\frac{1}{3}X + \frac{1}{2}Y\right) \quad [\because a = 1/3, b = ½]$$

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$= \frac{1}{9} \text{var}(X) + \frac{1}{4} \text{var}(Y)$

$$= \frac{36}{9} + \frac{36}{4} = 13$$

(iii).when a = ½, b = ½

$$\text{var}(T) = \text{var}(\frac{1}{2}X + \frac{1}{2}Y)$$

$$= \frac{1}{4} \text{var}(X) + \frac{1}{4} \text{var}(Y)$$

$$= \frac{1}{4}(36) + \frac{36}{4} = 18.$$

(iv) E(T) = (a+b) μ

For T to be an unbiased estimator of μ we should have E(T) = μ. This is possible for the following values of a and b.

(i) a = 0, b = 1

(ii) a = 1, b = 0

(iii) a = 1/2, b = ½

Example 1.4.12

Let $x_1, x_2 \ldots\ldots x_n$ be a random sample from a distribution having the p.d.f. $f(x,\theta) = \theta \ (1-\theta)^{x-1}$, $x = 1,2,3,\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ 0 \le \theta \le 1$.

Show that the sample mean $\bar{x}$ is an unbiased estimator of θ^{1}.

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Solution :

$$E(\overline{X}) = \sum_{x=1}^{\infty} x P(X=x).$$

$$= \sum_{x=1}^{\infty} x \theta (1-\theta)^{x-1}$$

$$= \theta (1-\theta)^{-1} \sum_{x=1}^{\infty} x (1-\theta)^{x}$$

$$= \frac{\theta}{(1-\theta)} [(1-\theta) + 2.(1-\theta)^2 + 3(1-\theta)^3 + ...]$$

$$= \frac{\theta}{(1-\theta)} (1-\theta) [(1 + 2(1-\theta) + 3(1-\theta)^2 + ...]$$

$$= \theta [1-(1-\theta)]^{-2}$$

$$E(\bar{x}) = \theta^{-1}$$

(i.e) Sample mean $\bar{x}$ is an unbiased estimate of the population mean θ^{-1}.

1.5 :SUFFICIENCY

An estimator is said to be **sufficient** for a parameter if it contains all the information in the sample regarding the parameter. More precisely if $T = t(x_1, x_2....x_n)$ is an estimator of a parameter θ based on a sample. $(x_1,x_2....x_n)$ of size n from the population with density $f(x,\theta)$ such that the conditional distribution of $(x_1,x_2....x_n)$ given T is independent of θ, then T is sufficient estimator for θ.

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Invariance property of Sufficient Estimator :

If T is a sufficient estimator for the parameter θ and if ψ (T) is a one to one function of T, then ψ (T) is sufficient for ψ (θ).

Example 1.5.1

Let $(x_1,x_2....x_n)$ be a random sample from a Bernoulli population with parameter

p, $0 < p < 1$.

(ie) x_i = 1 with probability p

= 0 with probability $q = 1 - p$.

Then $T = t(x_1,x_2....x_n) = x_1 +, x_2 +....+ x_n \sim B(n,p)$

Therefore $P(T = k) = nC_k\, p^k\, q^{n-k}$.

The conditional distribution of $(x_1,x_2....x_n)$ given T is

$P((x_1 \cap x_2 .... \cap x_n \mid T = k)$

$$= \frac{P(x_1 \cap x_2 .... \cap x_n T = k)}{P(T = k)}$$

$$= \left\{ \frac{p^k (1-p)^{n-k}}{nC_k P^k (1-p)^{n-k}} = \frac{1}{nC_k} \right.$$

Since this does not depend on p,

$T = \sum_{i=1}^{n} x_i$ is sufficient estimator for p.

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Example 1.5.2

Let $x_1, x_2 \ldots . x_n$ be a random sample from a population with p.d.f

$F(x,\theta) = \theta\, x^{\theta - 1}, 0 < x < 1, \theta > 0$

Show that $t_1 = \prod_{i-1}^{n} x_i$ is sufficient for θ.

Solution :

$$L(x, \theta) = \prod_{i-1}^{n} f(x_i, \theta)$$

$$= \theta^n \prod_{i-1}^{n} (x_i^{\theta - 1})$$

$$= \theta^n \left(\prod_{i-1}^{n} x_i \right)^{\theta} \frac{1}{\prod_{i=1}^{n} x_i}$$

$$= g(t_1, \theta).\ h(x_1, x_2 \ldots .. x_n) \quad \text{(say)}$$

Hence by factorization theorem

$t_i = \prod_{i=1}^{n} (x_i)$ is sufficient estimator for θ.

Example 1.5.3

Let $x_1, x_2, \ldots \ldots x_n$ be a random sample from a distribution with p.d.f

$.f(x.,\theta) = e^{-(x-\theta)}$ $\theta < x < \infty, \; -\infty < \theta < \infty$. Obtain the sufficient estimator for θ.

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Solution

$$\text{Here L} = \prod_{i=1}^{n} f(x_i,\theta)$$

$$= \prod_{i-1}^{n} e^{-(x_i-\theta)}$$

$$= e^{n\theta} \quad e^{--\sum_{i=1}^{n} x_i}$$

Let Y_1, $Y_2 \ldots . Y_n$ denote the order statistic of the random sample such that $Y_1 < Y_2 \ldots\ldots. < Y_n$.

The p.d.f of smallest observation Y_1 is given by

$g_1(y_1, \theta) = n\,[1-F(y_1)]^{n-1}\, f(y_1,\theta)$

Where F(.) is the distribution function corresponding to p.d. f f(.)

$$\text{Now } F(x) = \int_{\theta}^{x} e^{-(x-\theta)} dx$$

$$= [e^{-(x-\theta)}/-1]_{o}^{x}$$

$$= 1- e^{-(x-\theta)}$$

$$\therefore g_1(y_1,\theta) = n\,[1-(1-e^{-(y_1-\theta)})]^{n-1}$$

$$= n\,[e^{-(y_1-\theta)})]^{n-1}\, e^{-(y_1-\theta)}$$

$$= n\, e^{-(y_1-\theta)(n-1+1)}$$

$$= n\, e^{-(y_1-\theta)}, 0 < y_1 < \infty$$

$$= 0. \text{ otherwise}$$

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Thus the likelihood function of $(X_1, X_2 \ldots X_n)$ is expressed by

$$L = e^{n\theta} \exp(-\sum_{i=1}^{n} x_i)$$

$$= n \exp[-n(y_1 - \theta)] \exp(-\sum_{i=1}^{n} x_i) / \exp(-n.y_1)$$

$$= g_1 \min(x_i, \theta) \frac{\exp\left(-\sum_{i=1}^{n} x_i\right)}{n \exp(-n \min x_i)}$$

Hence by Fisher –Neymann criterion, the first order statistic,

$Y_1 = \min(X_1, X_2, .. X_n)$ is sufficient statistic for θ.

1.6 EFFICIENT ESTIMATORS

Let us consider two consistent estimators T_1 and T_2 of a certain parameter θ.

If var (T_1) < var (T_2) for all n, then T_1 is **more efficient** than T_2 for all sample size n.

Example 1.6.1

Prove that the sample mean is more efficient than the sample median, when the sample is taken from the normal population.

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Solution :

In the normal sample of size 'n' we have var $(\bar{x}) = \frac{\sigma^2}{n}$ and var (md) = $\frac{\pi\sigma^2}{2n}$ here $\bar{x}$ and median (md) are consistent estimators. Also var $(\bar{x})$ < var (md).

Therefore sample mean $\bar{x}$ is more efficient estimator for μ than the sample median for large samples.

Most efficient Estimator

Let us consider a class of consistent estimators for a parameter. Among the consistent estimators, there exists one whose sample variance is less than that of any such estimator, it is called **Most efficient Estimator**. Whenever such an estimator exists, it provides a criterion for measurement of efficiency of the other estimators.

Efficiency :

Definition

If T_1 is most efficient estimator with variance V_1 and T_2 is any other estimator with variance V_2, then the efficiency E of T_2 is defined as $E = \frac{V_1}{V_2}$

Example 1.6.2

In the normal sample for large size, sample mean $\bar{x}$ is the most efficient estimator of μ and Median (md) is any

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other estimator, then the efficiency E of median is

$$E = \frac{Var(\bar{x})}{Var(md)} = \frac{\sigma^2/n}{\pi\sigma^2/2n} = \frac{2}{\pi} = 0.637$$

1.7 Factorization Theorem due to Neyman.

The necessary and sufficient condition for a distribution to admit sufficient statistic is provided by the factorization theorem due to Neyman.

Statement :

$T = t(x)$ is sufficient for θ if and only if the joint density function L (say), of the sample values can be expressed in the form

$$L = g_\theta[(t(x)]\, h(x)$$

where $g_\theta[(t(x)]$ depends on θ and x only through the value of t(x) and h(x) is independent of θ.

Note 1:

Original sample $X = (x_1, x_2, \ldots\ldots\ldots\ldots x_n)$ is always a sufficient statistic.

Note 2: Koop man's form

The most general form of distributions admitting sufficient statistic is Koopman's form and is given by

$$L = L(x,\theta) = g(x).\, h(\theta) \exp[a(\theta)\, \psi\ (x)]$$

where $h(\theta)$ and $a(\theta)$ are functions of parameter θ only, and

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g(x) and ψ (x) are the functions of the sample observations only.

Fisher - Neyman Criterian :

A statistic $t_1 = t_1 (x_1, x_2, \ldots\ldots x_n)$ is sufficient estimator of parameter θ if and only if the likelihood function can be expressed as

$$L = \prod_{i=1}^{n} f(x_i, \theta)$$

$= g_1(t_1, \theta) \,.\, k(x_1, x_2, \ldots\, x_n)$

where $g_1 (t_1, \theta)$ is the p.d.f of the statistic t_1 and $k(x_1, x_2, \ldots x_n)$ is a function of sample observations only independent of θ.

1.8 :CRAMER RAO INEQUALITY

Theorem 1.8.(i)

If t is an unbiased estimator of γ (θ), a function of parameter θ, then

$$\text{Var}(t) \geq \frac{\left[\frac{d}{d\theta}\gamma(\theta)\right]^2}{E\left[\frac{\partial}{\partial\theta}\log L\right]^2} = \frac{[\gamma'(\theta)]^2}{I(\theta)}$$

Where I (θ) is the information on θ supplied by the sample. In other words Cramer inequality provides a lower bond $[(\gamma'(\theta)]^2 / I(\theta)$ to the variance of an unbiased estimator of γ (θ).

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Proof :

Assume that there is only a single parameter which is unknown. Consider the continuous random variable case. Regularity conditions for Cramer Rao inequality.

1. The parameter space ? is a non degenerate open interval on the real line $R^1(-\infty, \infty)$.
2. For almost all $x = (x_1, x_2, .. x_n)$ and for all $\theta \in$?

   $\frac{\partial}{\partial\theta}$ L (x, θ) exists, the exceptional set, if any, is independent of θ.
3. The range of the integration is independent of parameter θ, so that f(x, θ) is differentiable under integral sign

   If the range is not independent of θ and f is zero at the extremes of the range

   (ie) f(a, θ) = 0 = f (b, θ) then

$$\frac{\partial}{\partial\theta}\int_a^b f dx = \int_a^b \frac{\partial f}{\partial\theta} dx - f(a,\theta)\frac{\partial a}{\partial\theta} + f(b,\theta)\frac{\partial b}{\partial\theta}$$

$$\Rightarrow \frac{\partial}{\partial\theta}\int_a^b f\,dx = \int_a^b \frac{\partial f}{\partial\theta} dx$$

4. The conditions of uniform convergence of integrals are satisfied. So that differentiation under the integral sign is valid.

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1. $I(\theta) = E\left[\{\frac{\partial}{\partial\theta}\log L(x,\theta\}^2\right]$ exists and +ve for all

$\theta \in$?

Let X be a r.v following the p.d.f f(x, θ) and Let L be the likelihood function of the random sample $(x_1, x_2 \ldots. x_n)$ from this population.

Then $L = L(x,\theta) = \prod_{i=1}^{n} f(x_i,\theta)$

Since L is the joint p.d.f of $(x_1, x_2 \ldots x_n)$

$$\int L(x,\theta)dx = 1$$

Where $\int dx = \int\int \ldots\ldots \int dx_1 dx_2 \ldots. dx_n$

Differentiating w.r.to θ and using regularity conditions given above. We get

$$\int \frac{\partial}{\partial\theta} L dx = 1$$

$$\Rightarrow \int\left(\frac{\partial}{\partial\theta}\log L\right)L dx = 0$$

$$\Rightarrow E\left(\frac{\partial}{\partial\theta}\log L\right) = 0 \qquad \rightarrow (1)$$

Let $t = t(x_1\ x_2\ \ldots. x_n)$ be an unbiased estimator of $\gamma(\theta)$ such that

$$E(t) = \gamma(\theta) \Rightarrow \int tL dx = \gamma(\theta) \qquad \rightarrow (2)$$

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Diff. w.r. to θ

$$\int t.\frac{\partial L}{\partial\theta}dx = \gamma'(\theta)$$

$$\Rightarrow \int\left(t\frac{\partial}{\partial\theta}(\log L)\right)L dx = \gamma'(\theta)$$

$$\Rightarrow E\left(t\frac{\partial}{\partial\theta}\log L\right) = \gamma'(\theta) \qquad \rightarrow (3)$$

$$\text{cov}\left(t,\frac{\partial}{\partial\theta}\log L\right) = E\left(t\frac{\partial}{\partial\theta}(\log L)\right) - E(t).E\left(\frac{\partial}{\partial\theta}\log L\right)$$

$$= \gamma'(\theta) - E(t) \times 0$$

$$= \gamma'(\theta)$$

We have

$$[r(X,Y)]^2 \leq 1$$

$$[Cov(X,Y)]^2 \leq Var(X).Var(Y)$$

$$\left[\text{cov}(t,\frac{\partial}{\partial\theta}\log L)\right]^2 \leq Var(t).\text{var}(\frac{\partial}{\partial\theta}\log L)$$

$$[\gamma'(\theta)]^2 \leq \text{var}\, t.\left[E(\frac{\partial}{\partial\theta}\log L)^2\right]$$

$$\therefore \text{var}(t) \geq \frac{[\gamma'(\theta)]^2}{E\left[(\frac{\partial}{\partial\theta}\log L)^2\right]}$$

Which is known as Cramer Rao inequality.

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Corollary :

If t is an unbiased estimator of parameter θ (i.e) $E(t) = \theta \Rightarrow \gamma(\theta) = \theta \qquad \gamma'(\theta) = 1$

$$\therefore \ \text{var}(t) \geq \frac{1}{E\left[\left(\frac{\partial}{\partial\theta}\log L\right)^2\right]} = \frac{1}{I(\theta)}$$

Where $I(\theta) = E\left[\frac{\partial}{\partial\theta}(\log L)^2\right]$

$I(\theta)$ is called amount of information on θ supplied by the sample $(x_1, x_2, \ldots x_n)$ and

$I(\theta)$ is called as information limit to the variance of estimator $t = t(x_1\ x_2 \ldots\ldots x_n)$

Remarks :

1. An unbiased estimator t of $\gamma(\theta)$ for which. Cramer Rao lower bound attained is called, Minimum variance Bound (MVB) estimator.

2. $I(\theta) = E\left[\left(\frac{\partial}{\partial\theta}\log L\right)^2\right] = -E\left[\frac{\partial^2}{\partial\theta^2}\log L\right]$

Proof :

We know $E\left[\frac{\partial}{\partial\theta}\log L\right] = 0$

Also $\left(\frac{\partial^2}{\partial\theta^2}\log L\right)L = \frac{\partial}{\partial\theta}\left[\left(\frac{\partial}{\partial\theta}\log L\right).L\right] - \left(\frac{\partial}{\partial\theta}\log L\right)\frac{\partial L}{\partial\theta}$

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$$= \frac{\partial}{\partial\theta}\left[(\frac{\partial}{\partial\theta}\log L).L\right] - (\frac{\partial}{\partial\theta}\log L)^2.L$$

Integrating both sides with . r. to $\bar{x} = (x_1, x_2, \ldots x_n)$

$$E\left[\frac{\partial^2}{\partial\theta^2}\log L\right] = \frac{\partial}{\partial\theta}\text{E}\left[(\frac{\partial}{\partial\theta}\log L\right] - \text{E}\left[(\frac{\partial}{\partial\theta}\log L)\right]^2$$

$$= -\text{E}\left[(\frac{\partial}{\partial\theta}\log L)\right]^2$$

$$\Rightarrow I(\theta) = \text{E}\left[(\frac{\partial}{\partial\theta}\log L)\right]^2 = - E\left[\frac{\partial^2}{\partial\theta^2}\log L\right]$$

Example 1.8.1

If $X_1, X_2, \ldots\ldots X_n$ is a random sample from the poisson distribution with mean λ, find Cramer – Rao lower bound to the variance of any unbiased estimator of λ,

Solution :

$$f(x, \lambda) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$\log f(x, \lambda) = -\lambda + x\log\lambda - \log x!$$

$$\frac{\partial}{\partial\lambda}\log f(x, \lambda) = -1 + \frac{x}{\lambda}$$

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$$E\left[\frac{\partial}{\partial\lambda}\log f(x,\lambda)\right]^2 = E\left[-1+\frac{x}{\lambda}\right]^2$$

$$= \frac{1}{\lambda^2}E(x-\lambda)^2$$

$$= \frac{1}{\lambda^2}\ \mathrm{Var}(x)$$

$$= \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

Cramer – Rao Lower bound

$$= \frac{1}{nE\left[\frac{\partial}{\partial\theta}\log f(\theta)\right]^2}$$

$$= \frac{1}{n(\frac{1}{\lambda})} = \frac{\lambda}{n}.$$

1.9 RAO - BLACKWELL THEOREM :

Let X and Y be random variables such that E (Y) = μ and var (Y) $=\sigma_Y^2 > 0$

Let E [Y | X = x] = $\varphi(X)$ then

i. E [$\varphi(X)$] = μ

ii. Var [$\varphi(X)$] $\leq$ var (Y)

Proof

Let f_{XY} (x,y) be joint p.d.f random variable X and Y, $f_1(.)$ and $f_2(.)$ the marginal p.d.f of X and Y respectively

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and h [y | x] be the conditional p.d.f of Y for given X = x.

Such that $h(y/x) = \dfrac{f(x,y)}{f_1(x)}$

$$E(Y|X = x) = \int_{-\infty}^{\infty} y.h(y \mid x)dy$$

$$= \int_{-\infty}^{\infty} y.\frac{f(x,y)}{f_1(x)}dy$$

$$= \frac{1}{f_1(x)}\int_{-\infty}^{\infty} yf(x,y)dy$$

$$= \varphi(x)(\text{say}) \qquad \rightarrow(1)$$

$$\Rightarrow \int_{-\infty}^{\infty} yf(x,y)dy = \varphi(x)f_1(x) \qquad \rightarrow(2)$$

From (1) we see that the conditional distribution of Y given X = x does not depend on the parameter μ. Hence X is sufficient statistic for μ.

Now $E[\varphi(x)] = E[E(Y \mid X)]$

$$= E(Y)$$

$$= \mu$$

$\therefore$ part (i) is proved.

We have $\text{var}(Y) = E[Y-E(Y)]^2$

$$= E[Y-\mu]^2$$

$$= E[Y-\phi(x)+\phi(x)-\mu]^2$$

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and h [y | x] be the conditional p.d.f of Y for given X = x.

Such that $h(y/x) = \dfrac{f(x,y)}{f_1(x)}$

$$E(Y|X = x) = \int_{-\infty}^{\infty} y.h(y\,|\,x)dy$$

$$= \int_{-\infty}^{\infty} y.\frac{f(x,y)}{f_1(x)}dy$$

$$= \frac{1}{f_1(x)}\int_{-\infty}^{\infty} yf(x,y)dy$$

$$= \varphi(x)(\text{say}) \qquad \rightarrow (1)$$

$$\Rightarrow \int_{-\infty}^{\infty} yf(x,y)dy = \varphi(x)f_1(x) \qquad \rightarrow (2)$$

From (1) we see that the conditional distribution of Y given X = x does not depend on the parameter μ. Hence X is sufficient statistic for μ.

Now $E[\varphi(x)] = E[E(Y\,|\,X)]$

$= E(Y)$

$= \mu$

$\therefore$ part (i) is proved.

We have $\text{var}(Y) = E[Y-E(Y)]^2$

$= E[Y-\mu]^2$

$= E[Y-\phi(x) + \phi(x) - \mu]^2$

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$= E[Y - \phi(x)]^2 + E[\phi(x) - \mu]^2$

$$+2E[\{Y - \phi(x)\}\{\phi(x) - \mu\}] \rightarrow (3)$$

$E[\{Y - \phi(x)\}\{\phi(x) - \mu]$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (y - \phi(x))(\phi(x) - \mu) f(x,y)\,dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(y - \phi(x))(\phi(x) - \mu) f_1(x)\, h(y/x)dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\phi(x) - \mu]\left[\int_{-\infty}^{\infty}[y - \phi(x)]h(y/x)\right]dx$$

But $\int_{-\infty}^{\infty}[y - \phi(x)]\, h(y/x)\, dy = 0$

$\therefore E[(Y - \phi(x))(\phi(x) - \mu)] = 0$

subsitute in (3)

$Var(Y) = E[Y - \phi(x)]^2 + var[\phi(x)]$

$Var(Y) \geq var[\phi(x)]$ $\quad [\because E[Y - \phi(x)]^2 \geq 0]$

$\Rightarrow var(\phi(x)) \leq var(Y)$

Problems based on Cramer Rao inequality

Example 1.9.1

Given the p.d.f $f(x, \theta) = \pi[1 + (x - \theta)^2]^{-1}$

$-\infty < x < \infty$, $-\infty < \theta < \infty$. Show that the Cramer – Rao lower bound of variance of an unbiased estimator of θ is $\frac{2}{n}$, where n is the size of the sample from this distribution.

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Solution.

$\log f = -\log\pi - \log\,(1+(x-\theta\,)^2]$

$$\frac{\partial}{\partial\theta}\log f = 0 - \frac{1}{[1+(x-\theta\,)^2]}2(x-\theta\,)(-1)$$

$$= \frac{2(x-\theta\,)}{1+(x-\theta\,)^2}$$

$$E\left(\frac{\partial}{\partial\theta}\log f\right)^2 = \int_{-\infty}^{\infty}\frac{4(x-\theta\,)^2}{[1+(x-\theta\,)^2]^2}f(x,\theta\,)dx$$

$$=1/\pi\ \int_{-\infty}^{\infty}\frac{4(x-\theta\,)^2}{[1+(x-\theta\,)^2]^2}\frac{1}{[1+(x-\theta^{\,2}]^1}dx$$

$$=1/\pi\ \int_{-\infty}^{\infty}\frac{4(x-\theta\,)^2}{[1+(x-\theta\,)^2]^3}$$

Put $x - \theta\ = \tan\varphi$

$dx = \sec^2\ \varphi\ \ d\varphi$

$$E\left(\frac{\partial}{\partial\theta}\log f\right)^2 = \frac{2}{\pi}\int_0^{\infty}\frac{4\tan^2\varphi}{[1+\tan^2\varphi)^3}\sec^2\varphi d\varphi\quad d\varphi$$

$$= \frac{8}{\pi}\int_0^{\infty}\frac{\tan^2\varphi}{\sec^6\varphi}\sec^2\varphi\ \ d\varphi$$

$$= \frac{8}{\pi}\int_0^{\infty}\frac{\tan^2\varphi}{\sec^4\varphi}d\varphi$$

$$= \frac{8}{\pi}\int_0^{\infty}\frac{\sin^2\varphi}{\cos^2\varphi}\cos^4\varphi d\varphi$$

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$$=\frac{8}{\pi}\int_0^\infty \sin^2\varphi\cos^2\varphi \; d\varphi$$

$$=\frac{8}{\pi}\int_0^\infty (\cos^2\varphi-\cos^4\varphi)\; d\varphi$$

$$=\frac{8}{\pi}\left[\frac{1}{2},\frac{\pi}{2}-\frac{3}{4},\frac{1}{2},\frac{\pi}{2}\right]$$

$$=\frac{8}{\pi}\left[\frac{1}{2},\frac{\pi}{2}[1-3/4]\right]$$

$$=2(1/4)$$

$$=\frac{1}{2}$$

Hence Cramer – Rao lower bound is

$$=\frac{1}{nE(\frac{\partial}{\partial\theta}\log f)^2}$$

$$=\frac{1}{n(1/2)}=\frac{2}{n}$$

1.10. EXERCISE

1. What is meant by point estimation.
2. Define the following estimators and give an example for each.
   (i). Unbiased estimator (ii) Consistent Estimator
   (iii). Efficient estimator (iv) sufficient estimator.

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3. Distinguish between estimator and estimate.

4. Give an example of an estimator which is

   a. not unbiased but consistent.

   b. Unbiased and consistent

   c. Biased but not consistent.

   d. Unbiased, consistent, most efficient and also sufficient.

5. Show that $\frac{X+1}{n+2}$ is a biased estimate of the binomial parameter θ. Is this estimator asymtotically unbiased?

6. For the Geometric distribution

   $f(x,\theta) = \theta^{x}(1-\theta)^{x-1}$, $x = 1,2,3..........$, $0 < \theta < 1$

   Obtain an unbiased estimator of Y_θ.

7. Prove that an estimator of the mean of a normal population, sample mean is more efficient than the sample median.

8. Show that an unbiased estimator whose variance tends to zero as the sample size increased to infinity is consistent.

9. Obtain an unbiased estimator of p^2 for the Binomial probability distribution

   $f(x,p) = {}^{n}C_{x}\, p^{x} q^{n-x}$, $x = 0,1,2,...........n$, $0 < p < 1$.

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10. Let $x_1, x_2 \ldots x_n$ be a random sample of size 'n' drawn from a population with mean μ and variance σ^2. Obtain an unbiased estimator for μ^2.

11. Define a sufficient statistic. State and prove the factorization theorem on sufficiency.

12. State and prove Rao- Blackwell theorem.

13. State and prove Rao – Cramer in equality.

14. Let $x_1, x_2, \ldots x_n$ be a random sample from a population with p.d.f. $f(x,\theta) = \theta\, e^{-\theta x}$, $x > 0$, $\theta > 0$.

    Find Cramer – Rao lower bound for the variance of the unbiased estimator of θ.

15. Let (x_1, x_2) be a random sample from a Poisson distribution with parameter θ. Prove that $t = x_1 + 2x_2$ is not sufficient for θ.

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CHAPTER II

METHODS OF ESTIMATION AND CONFIDENCE INTERVAL

METHODS OF ESTIMATION

Good estimator can be obtained, but using some important methods. We shall briefly outline some of them.

Commonly used methods are

i. Method of maximum likelihood estimator

ii. Method of movements

iii. Method of least squares

iv. Method of minimum chi-square

v. Method of minimum variance

vi. Method of inverse probability

In the following section we shall discussed the first four methods briefly.

2.1 METHODS OF MAXIMUM LIKELIHOOD ESTIMATION

This method was formulated by C.F. Gauss

Likelihood function

Let $x_1,x_2,\ldots x_n$ be a random sample of size n from a population with density function (x_1,θ). Then the likelihood function of the ample values $x_1,x_2 \ldots .x_n$ denoted by L = L

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(θ) is their joint density function given by

$$L = f(x_1, \theta)\, f(x_2, \theta) \ldots\ldots f(x_n, \theta)$$

$$= \prod_{i=1}^{n} f(x_i, \theta)$$

L gives the relative likelihood that the random variable assume a particular set of values $x_1\ x_2, \ldots\ x_n$. For a given sample $x_1, x_2 \ldots x_n$ L becomes a function of the variable θ, the parameter. The principle of maximum likelihood consists in finding an estimator for the unknown parameter $\theta = (\theta_1, \theta_2 \ldots \theta_n)$ say which maximizes the likelihood function $L(\theta)$ for variations in parameter.

(ie) we wish to find $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \ldots \hat{\theta}_k)$ so that

$L(\hat{\theta}) > L(\theta) \quad \forall\, \theta \varepsilon\, \Theta.$

$L(\hat{\theta}) = \sup L(\theta) \quad \forall\, \theta \varepsilon\, \Theta.$

Thus if $\exists$ a function $\hat{\theta} = \hat{\theta}(x_1\ x_2 \ldots x_n)$ of the sample values which maximizes L for variations in θ, then $\hat{\theta}$ is to be taken as an estimator of θ. $\hat{\theta}$ is called MLE. Thus $\hat{\theta}$ is the solution if any of

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$$\frac{\partial L}{\partial \theta} = 0, \text{ and } \frac{\partial^2 L}{\partial \theta^2} < 0$$

Since L > 0, and log L is a non – decreasing function of L, L and log L attain their extreme values at the same value of

$\hat{\theta}$

Therefore $\frac{\partial L}{\partial \theta} = 0$ can be written as

$$\frac{1}{L} \cdot \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial}{\partial \theta} \log L = 0$$

If $\bar{\theta}$ is vector valued parameter, then $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \ldots \hat{\theta}_k)$

is given by solution simultaneous equations.

$$\frac{\partial}{\partial \theta_i} \log L = \frac{\partial}{\partial \theta_i} \log L\,(\theta_1, \theta_2 \ldots \theta_k) = 0 \quad i = 1,2,\ldots k.$$

Properties of maximum likelihood estimators

I The first and second order derivatives viz $\frac{\partial}{\partial \theta} \log L$ and $\frac{\partial^2}{\partial \theta^2} \log L$ exists and are continuous functions of. θ in a range R (including the true value θo) of the parameter for all most all x. For every θ in R.

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$$\left|\frac{\partial}{\partial\theta}\log L\right| < F_1(x) and \left|\frac{\partial^2}{\partial\theta^2}\log L\right| < F_2(x)$$

Where $F_1(x)$ and $F_2(x)$ are integrable over $(-\infty, \infty)$

ii. The third order derivative $\frac{\partial^3}{\partial\theta^3}\log L$ exists such that

$$\left|\frac{\partial^3}{\partial\theta^3}\log L\right| < M(x)$$

where $M(x) < k$ a +ve quantity.

iii For every θ in R

$$E\left(-\frac{\partial^2}{\partial\theta^2}\log L\right) = \int_{-\infty}^{\infty}\left(-\frac{\partial^2}{\partial\theta^2}\log L\right)L dx$$

is finite and non zero.

iv The range of integration is independent of θ.

But if the range of integration depends on θ, then $f(x,\theta)$ vanishes at the extremes depending on θ.

This assumption is to make the differentiation under the integral sign valid.

Theorem 2.1(i): (Cramer Rao – Theorem)

With probability approaching unity as $n \to \infty$ the likelihood equation $\frac{\partial}{\partial\theta}\log L = 0$ has a solution which converges in probability to the true value θ_o. In other words MLE's are consistent.

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Note : MLE's are always consistent estimation but not unbiased.

Example 2.1.1:

Consider a sample from normal population $N(\mu,\sigma^2)$, we have MLE of $\mu = \bar{x}$ which is both unbiased and consistent estimator of μ.

MLE $(\sigma^2) = s^2$ which is consistent but not an unbiased estimator of σ^2

Theorem 2.1.(ii): (Hazoor Bazar's theorem)

Any consistent solution of the likelihood equation provides a maximum of the likelihood with probability tending to unity as the sample size (n) tends to ∞.

Theorem 2.1. (iii): Asymptotic Normality of MLE's

A consistent solution of the likelihood equation is asymptotically normally distributed about the true value θo.

Thus $\hat{\theta}$ is asymptotically $N\left(\theta_0, \frac{1}{I(\theta_0)}\right)$ as $n \to \infty$

Note : Variance of MLE is given by

$$V(\hat{\theta}) = \frac{1}{I(\theta)} = \frac{1}{E\left(-\frac{\partial^2}{\partial\theta^2}\log L\right)}$$

Theorem 2.1.(iv) :

If MLE exists, it is the most efficient in the class of such estimators

Theorem 2.1. (v):

If a sufficient, estimator exists, it is a function of MLE.

Proof :

If t = t $(x_1,x_2,\ldots x_n)$ is a sufficient estimator of θ, then the likelihood function can be written as.

$$L = g\,(t, \theta).\, h(x_1, x_2\ldots x_n|\, t)$$

Where $g(t,\theta)$ is the density function of t and $h(x_1,x_2..x_n|\, t)$ is the density function of the sample, given t and is independent of θ.

$$\therefore \log L = \log g(t,\theta) + \log h(x_1, x_2 \ldots x_n / t)$$

Differentating w.r. to 'θ'.

$$\frac{\partial}{\partial\theta}\log L = \frac{\partial}{\partial\theta}\log\ g(t, \theta)$$

$$= \Psi\,(t, \theta) \quad \text{Say} \qquad \rightarrow (1)$$

Which is a function t and θ only

MLE is given by

$$\frac{\partial}{\partial\theta}\log L = 0 \Rightarrow \Psi\,(t, \theta) = 0$$

$$\therefore \hat{\theta} = \eta\,(t) = \text{some function of sufficient statistic}$$

$$\Rightarrow \hat{t} = \Psi\,(\theta) = \text{some function of MLE.}$$

Hence the proof.

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Note :

If $\frac{\partial}{\partial\theta}\log L$ expressed in the form of equation (1) as a function of statistic and parameter alone, then the statistic is regarded as a sufficient estimator for the parameter.

Theorem 2.1.(vi):

If for a given population with p.d.f f(x,θ) on MVB estimator T exists for θ then the likelihood equation will have a solution equal to the estimator T.

Proof :

Since T is an MVB estimator of θ, we have

$$\frac{\partial}{\partial\theta}\log L = \frac{T-\theta}{\lambda(\theta)} = (T-\theta)A(\theta)$$

MLE for θ is the solution of the likelihood equation

$$\frac{\partial}{\partial\theta}\log L = 0 \Rightarrow (T-\theta)\ A(\vartheta) = 0$$

$$\Rightarrow \hat{\theta} = T$$

Theorem 2.1. (vii) : (Invariance property of MLE)

If T is the MLE of θ and ψ (θ) is one to one function of θ, then ψ(t) is MLE of ψ (θ).

i. μ when σ^2 is known (ii) σ^2 when μ is known and (iii). Simultaneous estimation of μ and σ^2

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ii. Simultaneous estimate of μ and σ^2.

Solution : $X \sim N(\mu, \sigma^2)$ then

$$L = \prod_{i=1}^{n} f(x_i, \mu, \sigma^2)$$

$$L = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2\sigma^2}(x_i - \mu)^2\right]$$

$$L = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp\left(-\sum_{i=1}^{n} \frac{(x_i - \mu)^2}{2\sigma^2}\right)$$

$$\log L = -n \log(\sqrt{2\pi}\sigma) - \sum_{i=1}^{n} \frac{(x_i - \mu)^2}{2\sigma^2}$$

$$= \frac{n}{2}\log 2\pi - \frac{n}{2}\log \sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2$$

Case (i) When σ^2 is known likelihood equation of estimating μ is.

$$\frac{\partial}{\partial \mu}\log L = 0$$

$$\Rightarrow -\frac{1}{2\sigma^2} 2\sum_{i=1}^{n}(x_i - \mu)(-1) = 0$$

$$\Rightarrow \sum_{i=1}^{n}(x_i - \mu) = 0$$

$$\sum_{i=1}^{n} x_i - n\mu = 0$$

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$$\hat{\mu} = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$$

Hence MLE for μ is sample mean $\bar{x}$.

Case (ii) When μ is known, the likelihood equation for eliminating σ^2 is

$$\frac{\partial}{\partial \sigma^2} \log L = 0 \Rightarrow -\frac{n}{2\partial^2} + \frac{1}{2\partial^4} \sum_{i=1}^{n} (x_i - \mu)^2 = 0$$

$$\Rightarrow \text{n} - \frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2 = 0$$

$$\sigma^2 = \frac{\sum_{i=1}^{n} (x_i - \mu)^2}{n}$$

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu)^2$$

Case iii

The likelihood equation for simultaneous estimation of μ and σ^2 are

$$\frac{\partial}{\partial \mu} \log L = 0 \text{ and } \frac{\partial}{\partial \sigma^2} \log L = 0$$

$$\hat{\mu} = \bar{x} \text{ and } \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \hat{\mu})^2$$

$$\therefore \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

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$\hat{\sigma}^2 = s^2$, the sample variance

Note : $E(\hat{\mu}) = E(\bar{x}) = \mu$

$$E(\hat{\sigma}^2) = E(\sigma^2) \# \sigma^2$$

Hence MLE's need not necessarily be unbiased.

Note :

Since M.L.E is most efficient, we conclude in sampling from a normal population the sample mean $\bar{x}$ is the most efficient estimator of the population mean μ.

Example 2.1.3

Prove that the MLE of the parameter α of a population having density function

$\frac{2}{\alpha^2}(\alpha - x)$, $o < x < \alpha$ for a sample of unit size is 2x. x being the sample value. Show also that the estimate is biased.

Solution :

For a random sample of unit size (n - 1), the likelihood function is

$$L(\alpha) = f(x, \alpha) = \frac{2}{\alpha^2}(\alpha - x), \quad 0 < x < \alpha$$

$$\log L = \log 2 - 2 \log \alpha + \log (\alpha - x)$$

Likelihood equation gives

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$$\frac{d}{d\alpha}\log L = \frac{-2}{\alpha} + \frac{1}{\alpha - x}(1) = 0$$

$$\frac{1}{\alpha - x} = \frac{2}{\alpha}$$

$$\alpha = 2\alpha - 2x$$

$$\alpha = 2x$$

Therefore M.L.E of α is $\hat{\alpha} = 2x$

To show that estimate is biased :

$$E(\hat{\alpha}) = E(2x)$$

$$= 2E(x)$$

$$= 2\int_0^\alpha x \times f(x,\alpha)dx$$

$$= 2\int_0^\alpha x \times \frac{2}{\alpha^2}(\alpha - x)dx$$

$$= \frac{4}{\alpha^2}\left(\alpha\frac{x^2}{2} - \frac{x^3}{3!}\right)_o^\alpha$$

$$= \frac{4}{\alpha^2}\left(\frac{\alpha^3}{2} - \frac{\alpha^3}{3}\right)$$

$$= \frac{4}{\alpha^2}\left(\frac{\alpha^3}{6}\right)$$

$$= \frac{2}{3}\alpha$$

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Since $E\left(\hat{\alpha}\right) \neq \alpha$, $\hat{\alpha} = 2x$ is not an unbiased estimate of α.

Example : 2.1.4

a. Find MLE for the parameter λ of a Poisson distribution on the basis of a sample of size n. Also fine its variance.

b. Show that sample mean $\bar{x}$, is sufficient for estimating the parameter λ of poisson distribution.

Solution :

The p.d.f of Poisson distribution with parameter λ is given by

$$P(X = x) = f(x, \lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad x = 0,1,2\ldots.$$

Likelihood function of random sample $x_1, x_2 \ldots. x_n$ of a n observations from this population is

$$L = \prod_{i=1}^{n} f(x_i, \lambda)$$

$$= \frac{e^{-\lambda}\lambda^{x_1}}{x_1!} \cdot \frac{e^{-\lambda}\lambda^{x_2}}{x_2!} \cdots\cdots \frac{e^{-\lambda}\lambda^{x_n}}{x_n!}$$

$$L = \frac{e^{-n\lambda}\lambda^{\sum_{i=1}^{n} x_i}}{x_1! . x_2! \ldots.. x_n!}$$

$$\log L = -n\lambda + \sum_{i=1}^{n} x_i \log\lambda - \sum_{i=1}^{n} \log(x_i!)$$

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Likelihood equation for eliminating λ is $\frac{\partial}{\partial\lambda}\log\lambda = 0$

$$\Rightarrow -n + \frac{1}{\lambda}n\bar{x} = 0$$

$$\frac{1}{\lambda}n\bar{x} = n$$

$$\lambda = \bar{x}$$

$\therefore$ MLE of λ is $\bar{x}$

The variance of the estimate is given by

$$\frac{1}{V(\hat{\lambda})} = E\left[-\frac{\partial^2}{\partial\lambda^2}(\log L)\right]$$

$$= E\left[-\frac{\partial}{\partial\lambda}\frac{\partial}{\partial\lambda}(\log L)\right]$$

$$= E\left[-\frac{\partial}{\partial\lambda}(-n + \frac{n\bar{x}}{\lambda})\right]$$

$$= E\left[\frac{n\bar{x}}{\lambda^2}\right]$$

$$= \frac{n}{\lambda^2}E(\bar{x})$$

$$\frac{1}{Var(\hat{\lambda})} = \frac{n}{\lambda^2}\lambda = \frac{n}{\lambda}$$

$$\therefore Var((\hat{\lambda}) = \frac{\lambda}{n}$$

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b. For Poisson distribution with parameter λ, we have

$$\frac{\partial}{\partial\lambda}\log L = -n + n\frac{\bar{x}}{\lambda}$$

$$= \left(\frac{\bar{x}}{\lambda} - 1\right) n$$

$$= \psi \; y(\bar{x}, \lambda) \text{ a function of } \bar{x} \text{ and } \lambda \text{ only.}$$

$\bar{x}$ is sufficient for estimating λ.

Example : 2.1.5

Let $x_1, x_2, \ldots x_n$ denote random sample size 'n' from a uniform population with p.d.f $f(x,\theta) = 1$, $\theta - \frac{1}{2} \leq x \leq \theta + \frac{1}{2}$, $-\infty < \theta < \infty$

Solution :

Here $L = L(\theta: x_1, x_2 \ldots x_n) = 1 \quad \theta - \frac{1}{2} \leq x_i \leq \theta + \frac{1}{2}$

$= 0$, elsewhere

If $x_{(1)}, x_{(2)}, \ldots x_{(n)}$ is the ordered sample then

$\theta - \frac{1}{2} \leq x_{(1)} \leq x_{(2)} \leq \ldots \ldots \leq x(n) \leq \theta + \frac{1}{2} = 0$

elsewhere

If $x_{(1)}, x_{(2)} \ldots x_{(n)}$ is the ordered sample

Then $\theta - \frac{1}{2} \leq x_{(1)} \leq x_{(2)} \leq \ldots \ldots \leq x(n) \leq \theta + \frac{1}{2}$

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This L attains the maximum if

$$\theta - \frac{1}{2} \le x_{(1)} \wedge x_{(n)} \le \theta + \frac{1}{2}$$

$$\Rightarrow \theta \le x_{(1)} + \frac{1}{2} \wedge x_{(n)} - \frac{1}{2} \le \theta$$

Hence every statistic $t = t(x_1\ x_2 \ldots x_n)$ such that

$$X_{(n)} - \frac{1}{2} \le t(x_1\ x_2 \ldots . x_n) \le x_{(1)} + \frac{1}{2}$$

Which provides M.L.E. for θ

Example : 2.1.6

Find the M.L.E. of parameters α and λ (λ being large), of the distribution.

$$f(x_1 : \alpha, \lambda) = \left(\frac{1}{\Gamma(\lambda)}\right) \left[\frac{\lambda}{\alpha}\right]^{\lambda} e^{-\lambda x/\alpha} . x^{\lambda-1} \quad 0 \le x < \infty, \lambda > 0$$

You my use. that for large values of λ

$$\Psi(\lambda) = \frac{\partial}{\partial\lambda} \log \Gamma(\lambda) = \log \lambda - \left[\frac{1}{2\lambda}\right] \text{ and}$$

$$\Psi'(\lambda) = \frac{1}{\lambda} + \frac{1}{2\lambda^2}$$

Solution :

Let $x_1\ x_2 \ldots . x_n$ be a random sample of size n from the given population.

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Then

$$L = \prod_{i=1}^{n} f(x, \alpha, \lambda)$$

$$L = \left(\frac{1}{\Gamma(\lambda)}\right)^{n} \cdot \left(\frac{\lambda}{\alpha}\right)^{n\lambda} e^{-\frac{\lambda}{\alpha}\sum_{i=1}^{n} x_i} \prod_{i=1}^{n} (x_i^{\lambda-1})$$

$$\log L = -n \log \Gamma(\lambda) + n\lambda (\log \lambda - \log \alpha) - \frac{\lambda}{\alpha}\sum_{i=1}^{n} x_i + (\lambda - 1)\sum_{i=1}^{n} \log x_i$$

If G is the geometric mean of $x_1, x_2 \ldots.. x_n$ then

$$\log G = \frac{1}{n}\sum_{i=1}^{n} \log x_i$$

$$n. \log G = \sum_{i=1}^{n} \log x_i$$

$$\therefore \log L = -n \log \Gamma(\lambda) + n\lambda (\log \lambda - \log \alpha) - \left(\frac{\lambda}{\alpha}\right)\sum_{i=1}^{n} x_i + (\lambda - 1)\, n \log G$$

Where G is independent of λ and α

The likelihood equations for simultaneous estimation of α

are $\frac{\partial}{\partial \alpha} \log L = 0$ and $\frac{\partial}{\partial \lambda} \log L = 0$

$$\frac{\partial}{\partial \alpha} \log L = 0 \Rightarrow -\frac{n\lambda}{\alpha} + \frac{\lambda}{\alpha^2} n\bar{x} \quad = 0$$

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$$\frac{\lambda}{\alpha}(-n+\frac{n\bar{x}}{\alpha}) = 0$$

$$\frac{n\bar{x}}{\alpha} = n$$

$$\alpha = \bar{x}$$

$$\hat{\alpha} = \bar{x}$$

$$\frac{\partial}{\partial\lambda}\log L = 0$$

$$\Rightarrow -n\frac{\partial}{\partial\lambda}\log\ \Gamma(\lambda) + n\left[\lambda(\frac{1}{\lambda}) + (\log\lambda - \log\alpha)(1)\right]$$

$$-\frac{1}{\alpha}n\bar{x} + n\log G = 0$$

$$\Rightarrow -n(\log\lambda - \frac{1}{2\lambda}) + [1 + \log\lambda - \log\alpha] - \frac{n}{\alpha}\bar{x} + n\log G = 0$$

$$-n\log\lambda + \frac{n}{2\lambda} + n + n\log\lambda - n\log\alpha - \frac{n}{\alpha}\bar{x} + n\log G = 0$$

$$\frac{1}{2\lambda} + (1 - \log\alpha - \frac{\bar{x}}{\alpha} + \log G) = 0$$

We have $\alpha = \bar{x}$.

Therefore $\frac{1}{2\lambda} + (1 - \log\alpha + \frac{\bar{x}}{\bar{x}} + \log G) = 0$

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$$\frac{1}{2\lambda}+(1-\log\alpha-1+\log G)=0$$

$$\frac{1}{2\lambda}+(\log G-\log\bar{x})=0$$

$$1+2\lambda(\log G-\log\bar{x})=0$$

$$1-2\lambda\log\left(\frac{\bar{x}}{G}\right)=0$$

$$2\lambda\log\left(\frac{\bar{x}}{G}\right)=1$$

$$\lambda=\frac{1}{2\log\left(\frac{\bar{x}}{G}\right)}$$

Therefore $\hat{\lambda}=\dfrac{1}{2\log\left(\frac{\bar{x}}{G}\right)}$

Therefore M.L.E's for α and λ are given by

$$\hat{\alpha}=\bar{x} \text{ and } \hat{\lambda}=\frac{1}{2\log(\bar{x}/G)}$$

Example : 2.1.7

In sampling from a power series distribution with p.d.f $f(x,\theta) = a_x\theta^x/\psi(\theta)$ x= 0,1,2,.... where a_x may be zero for a some x show that MLE of θ is a root of the equation $\bar{x}=\dfrac{\theta\psi'(\theta)}{\psi(\theta)}=\mu(\theta)$ where $\mu(\theta)=E(\alpha)$

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Solution :

Likelihood function is given by

$$L = \prod_{i=1}^{n} f(x_i, \theta)$$

$$= \prod_{i=1}^{n} \left[\frac{a_{x_i}\theta^{x_i}}{\psi(\theta)}\right]$$

$$= \prod_{i=1}^{n} a_{x_i} \times \frac{\theta^{\Sigma x_i}}{[\psi(\theta)]^n}$$

$$\Rightarrow \log L = \sum_{i=1}^{n} \log a_{x_i} + \sum_{i=1}^{n} x_i \log\theta \text{ -n } \log \psi(\theta)$$

Likelihood function for estimating θ gives

$$\frac{\partial}{\partial\theta}\log L = 0 \Rightarrow \frac{1}{\theta}\sum_{i=1}^{n} x_i - \frac{n}{\psi(\theta)}\psi'(\theta) = 0$$

$$\frac{\sum_{i=1}^{n} x_i}{\theta} = \frac{n}{\psi(\theta)}\psi'(\theta)$$

$$\bar{x} = \frac{\theta\psi'(\theta)}{\psi(\theta)} = \mu(\theta) \text{ (Say)} \qquad \text{---- (1)}$$

Hence M.L.E of θ is root of equation (1)

We have

$$E(x) = \sum_{x=0}^{\infty} x f(x,\theta)$$

$$= \sum_{x=0}^{\infty} x \left[\frac{a_x\theta^x}{\psi(\theta)}\right]$$

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$$\sum_{x=0}^{\infty} f(x,\theta) = 1$$

$$\Rightarrow \sum_{x=0}^{\infty} \frac{a_x \theta^x}{\psi(\theta)} = 1$$

$$\Rightarrow \sum_{x=0}^{\infty} a_x \theta^x = \psi(\theta)$$

Diff w.r 'θ'

$$\sum_{x=0}^{\infty} a_x x \theta^{x-1} = \psi'(\theta)$$

$$\sum_{x=0}^{\infty} a_x \frac{x\theta^x}{\psi(\theta)} = \frac{\theta\psi'\theta}{\psi(\theta)}$$

$\therefore$ E(x) $= \mu(\theta) = \bar{x}$

Example : 2.1.8

a. Let $x_1, x_2, \ldots x_n$ be a random sample from the uniform distribution with p.d.f

$$f(x, \theta) = \frac{1}{\theta}, 0 < x < \infty, \theta > 0$$

$$= 0, \text{ elsewhere}$$

Obtain M.L.E. for θ.

a. Obtain M.L.E for α and β for rectangular population.

$$f(x, \alpha, \beta) = \frac{1}{\beta - \alpha}, \alpha < x < \beta$$

$$= 0, \quad \text{else where,}$$

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Solution :

$$L = \prod_{i=1}^{n} f(x_i, \theta)$$

$$= \frac{1}{\theta} \cdots\cdots \frac{1}{\theta} = \left(\frac{1}{\theta}\right)^n \qquad \cdots\cdots\cdots\cdots\cdots \qquad (*)$$

$\log L = -n \log \theta$

Likelihood equation

$$\frac{\partial}{\partial\theta} \log L = 0 \text{ gives}$$

$$\frac{\partial}{\partial\theta} (-\, n \log \theta) = 0$$

(i.e) $-\dfrac{n}{\theta} = 0$

$\hat{\theta} = \infty$ (absurd result)

In this case we locate M.L.E as follows. We have to choose θ so that L in (*) is maximum. Now L is maximum if θ is minimum.

Let $x_{(1)}, x_{(2)}, \ldots\ldots\ldots\ldots\ldots x_{(n)}$ be the ordered sample of n independent observations from the given population so that

$$0 \le x_{(1)} \le x_{(2)} \ldots \le x_{(n)} \le \theta \Rightarrow \theta \ge x_{(n)}$$

Since minimum value of θ consistent with sample is $x_{(n)}$ the largest sample. observation $\hat{\theta} = x_{(n)}$

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Therefore M.L.E for $\theta = x_{(n)}$ = The largest sample observation.

(b). Here $L = (\frac{1}{\beta - \alpha})^n$ .... (**)

$\log L = -n \log(\beta - \alpha)$

The likelihood equation for α and β give

$$\frac{\partial}{\partial \alpha} \log L = 0 \Rightarrow \frac{n}{\beta - \alpha} = 0$$

$$\frac{\partial}{\partial \beta} \log L = 0 \quad - \frac{n}{\beta - \alpha} = 0$$

Each of these equations gives $\beta - \alpha = \infty$ which is absurd result

Now L in (**) is maximum if $(\beta - \alpha)$ is minimum (i.e) if β takes the minimum possible value and α takes the maximum possible value.

Let $x_{(1)} \leq x_{(2)} \ldots\ldots\ldots\ldots \leq x_n$ be an ordered random sample, then

$\alpha \leq x_{(1)} \leq x_{(2)} \ldots\ldots\ldots\ldots \leq x_{(n)} \leq \beta$

Thus $\beta \geq x_{(n)}$ and $\alpha \leq x_{(1)}$

Hence minimum possible value of β consistent with the sample is $x_{(n)}$ and maximum possible value of a consistent with the sample is $x_{(1)}$. Hence L is maximum

if $\beta = x_{(n)}$ and $\alpha = (x_{(1)}$

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Therefore M.L.E for α and β are given by

$\hat{\alpha} = x_{(1)}$ = The smallest sample observation

$\hat{\beta} = x_{(n)}$ = The largest sample observation.

Example : 2.1.9

Find M.L.E of θ in $f(x,\theta) = (1+\theta)\, x^{\theta} \quad 0 < x < 1$ based on independent sample of size n. Estimate whether the estimate is sufficient for θ.

Solution :

$$L(x, \theta) = \prod_{i=1}^{n} f(x_i, \theta)$$

$$= (1+\theta)^n \prod_{i=1}^{n} (x_i^{\theta})$$

Take log,

$$\log L = n \log (1+\theta) + \theta \sum_{i=1}^{n} \log x_i$$

$$\frac{\partial}{\partial\theta} \log L = \frac{n}{1+\theta} + \sum_{i=1}^{n} \log x_i$$

$$\frac{\partial}{\partial\theta} \log L = 0$$

$$\Rightarrow \frac{n}{1+\theta} + \sum_{i=1}^{n} \log x_i = 0$$

$$n + (1+\theta) \sum_{i=1}^{n} \log x_i = 0$$

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$$n + \sum_{i=1}^{n} \log x_i + \theta \sum_{i=1}^{n} \log x_i = 0$$

$$\theta \sum_{i=1}^{n} \log x_i = -n - \sum_{i=1}^{n} \log x_i$$

$$\theta = -\frac{n}{\sum_{i=1}^{n} \log x_i} - 1$$

$$= \frac{-n}{\log\left(\prod_{i=1}^{n} x_i\right)} - 1$$

Also

$$L(x,\theta) = (1+\theta)^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1} \left(\prod_{i=1}^{n} x_i\right)$$

By factorization theorem,

$T = \left(\prod_{i=1}^{n} x_i\right)$ is a sufficient statistic for θ . Also $\hat{\theta}$ is one to onefunction of sufficient statistic $\left(\prod_{i=1}^{n} x_i\right)$ is also sufficient for θ.

Properties of MLE

1. The ML estimator of a parameter is a consistent estimator

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2. The ML estimator is asymptotically normally distributed with mean θ and variance $\frac{1}{R^2(\theta)}$, where

$$R^2(\theta) = E\left[\frac{\partial}{\partial\theta} \log L(x,\theta)^2 \right]$$

3. The ML estimators are most efficient and best asymptotically normal estimator.
4. If there exists a sufficient estimator for paramatoer θ, then the ML estimator of θ is a function μ.
5. The ML estimators are not necessarily unbiased. But quiet often an ML estimator multiplied by a constant is unbiased.
6. ML estimators possess the property of invariance that is if $\hat{\theta}$ is a MLE of θ then $u(\hat{\theta})$ is also the ML estimator of u(θ).

Example : 2.1.10

Obtain the M.L.Es of α and β for a random sample from the exponential population $f(x, \alpha,\beta) = y_0\, e^{-\beta(x-\alpha)}$

$\alpha \le x \le \infty, \beta > 0$

y_0 being a constant.

To find constant y_0

Since total area under the probability is unity

$$\int_0^\infty y_o e^{-\beta(x-\alpha)}\,dx = 1$$

$$\Rightarrow y_o \int_0^\infty e^{-\beta(x-\alpha)}\,dx = 1$$

$$\Rightarrow y_o \left[\frac{e^{-\beta(x-\alpha)}}{-\beta}\right] 1 .$$

$$\Rightarrow - \frac{y_o}{\beta}(0-1) = 1$$

$$\Rightarrow y_0 = \beta$$

$$\therefore \quad f(x;\alpha,\beta) = \beta \; e^{-\beta(x-\alpha)}, \quad \alpha \le x < \infty$$

If $x_1, x_2, \ldots . x_n$ is a random sample of size n from this population then

$$L = \prod_{i=1}^{n} f(x_i, \alpha, \beta)$$

$$L = \beta^n \exp\left\{-\beta \left(\sum_{i=1}^{n} x_i - \alpha\right)\right\}$$

$$\log L = n \log \beta - \beta \left(\sum_{i=1}^{n} (x_i - \alpha)\right)$$

$$= n \log \beta - \beta \left(\sum_{i=1}^{n} x_i - n\alpha\right)$$

$$= n \log \beta - \beta \,[n \bar{x} - n\alpha]$$

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$\log L = n \log \beta - n\beta\,[\bar{x} - \alpha\,]$ ... (*)

The likelihood equations for estimating α and β are

$$\frac{\partial}{\partial\alpha}\log L = 0 \text{ and } \frac{\partial}{\partial\beta}\log L = 0$$

$$\frac{\partial}{\partial\alpha}\log L = 0 + n\beta$$

$$n\beta = 0$$

$\Rightarrow\ \beta = 0$ ....(1)

which is inadmissible.

Also $\dfrac{\partial}{\partial\beta}\log L = 0$

$\Rightarrow\ \dfrac{n}{\beta} - n\,(\bar{x} - \alpha\,) = 0 \rightarrow$ ...(2)

Substitute (1) in (2)

$\infty - n(\bar{x} - \alpha) = 0$

$\alpha = \infty$ nugatory result.

We try to locate M.L.E.s for α and β by maximizing L directly.

L is maximum $\Rightarrow$ log L is maximum.

From (*) log L is maximum if $\bar{x} - \alpha$ is minimum for any value of β . $\bar{x} - \alpha$ is minimum is possible if α is maximum.

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If $x_{(1)}, x_{(2)} \dots\dots\dots x_{(n)}$ is ordered sample from this population then $\alpha \le x_{(1)} \le x_{(2)} \dots\dots\dots\dots \le x_{(n)} < \infty$ so that the maximum value of α consistent with the sample is $x_{(1)}$, the smallest observation. Therefore $\hat{\alpha} = x_{(1)}$

Substitute the value of $\hat{\alpha}$ in (2)

$$\frac{n}{\beta} - n(\bar{x} - \hat{\alpha}) = 0$$

$$\Rightarrow \frac{1}{\beta} = \bar{x} - \hat{\alpha}$$

$$\Rightarrow \frac{1}{\beta} = \bar{x} - x_{(1)}$$

$$\Rightarrow \hat{\beta} = \frac{1}{\bar{x} - x_{(1)}}$$

Hence M.L.E.s for α and β are given by $\hat{\alpha} = x_{(1)}$ and

$$\hat{\beta} = \frac{1}{\bar{x} - x_{(1)}}$$

Example : 2.1.11

Let $(x_1, x_2 \dots\dots\dots\dots\dots\dots\dots x_n)$ be a random sample from the point binomial distribution with parameter p find M.L.E. of p.

Solution :

The probability function of the point binomial

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distribution is $p(x) = p^x q^{n-x}$. The likelihood function is given by

$$L = \prod_{i=1}^{n} p^x q^{n-x}$$

$$= p^{\sum_{i=1}^{n} x_i} q^{n-\sum_{i=1}^{n} x_i}$$

$$\log L = \sum_{i=1}^{n} x_i \log p + (n- \sum_{i=1}^{n} x_i) \log q$$

$$= n\bar{x} \log p + (n-n\bar{x}) \log q$$

$$\log L = n\bar{x} \log p + (n-n\bar{x}) \log (1-p)$$

The likelihood equation for parameter p is

$$\frac{\partial}{\partial p} \log L = 0$$

$$\Rightarrow \frac{n\bar{x}}{p} + \frac{n(1-\bar{x})}{(1-p)}(-1) = 0$$

$$\Rightarrow \frac{\bar{x}}{p} - \frac{(1-\bar{x})}{(1-p)} = 0$$

$$\Rightarrow (1-p)\ \bar{x} - p\,(1-\bar{x}) = 0$$

$$\Rightarrow \bar{x} - p\bar{x} - p + p\bar{x} = 0$$

$$\Rightarrow \hat{p} = \bar{x}$$

Also $\frac{\partial^2}{\partial p^2} \log L = -\ \frac{n\bar{x}}{p^2} - \frac{n(1-\bar{x})}{(1-p)^2}$

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$$\left(\frac{\partial^2}{\partial p^2}\log L\right)_{at\ \hat{p}=\bar{x}} = -\frac{n\bar{x}}{\bar{x}^2} - \frac{n(1-\bar{x})}{(1-\bar{x})^2}$$

$$= \frac{-n[\bar{x}(1-\bar{x})^2 + \bar{x}(1-\bar{x})]}{\bar{x}^2(1-\bar{x})^2}$$

$$= \frac{-n\bar{x}(1-\bar{x})}{\bar{x}^2(1-\bar{x})^2}[1-\bar{x}+\bar{x}]$$

$$= \frac{-n\bar{x}(1-\bar{x})}{\bar{x}^2(1-\bar{x})^2} < 0$$

Hence M.L.E of p is $\bar{x}$

Example : 2.1.12

Derive M.L.E. for θ , given a random sample from the population , with density function.

$f(x,\theta) = \{ e^{-(x-\theta)}$ for $x \geq \theta$

$= 0$ otherwise

Solution

$$L = \prod_{i=1}^{n} f(x_i\theta)$$

$$= \prod_{i=1}^{n} e^{-(x_j-\theta)}$$

$$= \prod_{i=1}^{n} e^{\theta} .\ e^{-x_i}$$

$$L = e^{n\theta}\ e^{-\sum_{i=1}^{n} x_i}$$

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$$\log L = n\theta - \sum_{i=1}^{n} x_i$$

The likelihood equation for θ. Is

$$\frac{\partial}{\partial\theta} \log L = 0$$

Therefore n = 0

Thus the likelihood equations fail to give us valid estimate of θ. Since n $\neq 0$, we choose θ such that L is maximum where θ takes the smallest value. Suppose the sample values are arranged in $x_{(1)}, x_{(2)}, \ldots\ldots\ldots x_{(n)}$ then $x_{(1)}$ will be the smallest value of the sample. ∴ M.L.E of θ is

$$\hat{\theta} = x_{(1)}$$

Example : 2.1.13

A random sample of size 'n' drawn is from the population

$$f(x) = \frac{1}{b} \exp\left(\frac{-(x-a)}{b}\right), \; a \leq x \leq \infty$$

$$= 0 \quad \text{elsewhere}$$

Where $-\infty < a < \infty$ and $o < b < \infty$ Find M.L.E's for a and b.

Solution :

$$L = \prod_{i=1}^{n} f(x_i, a, b)$$

$$= \prod_{i=1}^{n} \frac{1}{b} \exp\left\{-\frac{x-a}{b}\right\}$$

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$$L = \left(\frac{1}{b}\right)^n \exp\left\{-\frac{1}{b}\sum_{i=1}^{n} x_i - a)\right\} \quad ..... \quad 1$$

$$\log L = - n \log b - \frac{1}{b}\sum_{i=1}^{n}(x_i - a)$$

Likelihood equation for a and b are

$$\frac{\partial}{\partial a}\log L = 0 \text{ and } \frac{\partial}{\partial b}\log L = 0$$

$$\frac{\partial}{\partial a}\log L = 0$$

$$\Rightarrow \frac{n}{b} = 0$$

$\Rightarrow$ b = ∞ which is inadmissible

$$\frac{\partial}{\partial b}\log L = 0$$

$$\Rightarrow -\frac{n}{b} + \frac{1}{b^2}\sum_{i=1}^{n}(x_i - a) = 0$$

$$\frac{1}{b}\sum_{i=1}^{n}(x_i - a) = \text{n}$$

Substituting b = ∞ we have n = 0, which is absurd result. Hence likelihood equations fails us to give the M.L.E for a and b.

From equation (1), L is maximum when $\sum_{i=1}^{n}(x_i - a)$ is minimum. This is possible when a must be larger subject to

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the condition $a \le x_{(1)} < x_{(2)} \ldots\ldots \le x_{(n)} \le \infty$, where $x_{(1)} \le x_{(2)} \ldots\ldots \le x_{(n)}$ is the ordered sample, of size n.

$a = \min\{x_{(1)}, x_{(2)} \ldots\ldots\ldots\ldots x_{(n)}\} = x_{(1)}$

$\hat{a} = x_{(1)}$ = smallest sample observation

Substitute $\hat{a} = x_{(1)}$ in equation ....(1)

$$L = \frac{1}{b^n} \exp\left\{-\frac{1}{b}\sum_{i=1}^{n}(x_i - x_{(1)})\right\}$$

$$\log L = -n \log b - \frac{1}{b}\sum_{i=1}^{n}(x_i - x_{(1)})$$

$$\frac{\partial}{\partial b}\log L = -\frac{n}{b} + \frac{1}{b^2}\sum_{i=1}^{n}(x_i - x_{(1)})$$

$$\frac{\partial}{\partial b}\log L = 0$$

$$\Rightarrow -\frac{n}{b} + \frac{1}{b^2}\sum_{i=1}^{n}(x_i - x_{(1)}) = 0$$

$$\frac{1}{b}\sum_{i=1}^{n}(x_i - x_{(1)}) = n$$

$$b = \frac{1}{n}.\sum_{i=1}^{n}(x_i - x_{(1)})$$

Therefore M.L.E for b, is

$$\hat{b} = \frac{1}{n}\sum_{i=1}^{n}(x_i - x_{(1)})$$

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Example : 2.1.14

Find M.L.E of θ given that the p.d.f $f(x, \theta) = \theta x^{\theta-1}$, $0 < x < \infty, 0 < \theta < \infty$

Solution :

$$L(x, \theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1}$$

$$= \theta x_1^{\theta-1} . \theta x_2^{\theta-1} \ldots\ldots\ldots \theta x_n^{\theta-1}$$

$$= \theta^n \prod_{i=1}^{n} x_i^{\theta-1}$$

$$\log L = n \log \theta + (\theta - 1) \left[\sum_{i=1}^{n} \log x_i\right]$$

Likelihood equation for estimating θ is

$$\frac{\partial}{\partial \theta} \log L = 0$$

$$\Rightarrow \frac{n}{\theta} + \sum_{i=1}^{n} \log x_i = 0$$

$$\frac{n}{\theta} = -\sum_{i=1}^{n} \log x_i$$

$$\theta = \frac{-n}{\sum_{i=1}^{n} \log x_i}$$

$$\theta = \frac{-n}{\log(x_1, x_2 \ldots x_n)}$$

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M.L.E. of θ is

$$\hat{\theta} = \frac{-n}{\log(x_1, x_2 \ldots x_n)}$$

Example : 2.1.15

Find M.L.E of the parameter β for the p.d.f

$$f(x,\beta) = \frac{1}{\beta}, -\beta < x < 0, 0 < \beta < \infty$$

$$L(x,\beta) = \prod_{i=1}^{n}\left(\frac{1}{\beta}\right)$$

$$= \left(\frac{1}{\beta}\right)^n = \frac{1}{\beta^n}$$

$$\log L = -n \log \beta.$$

Likelihood equation

$$\frac{\partial}{\partial \beta} \log L = 0$$

$$\Rightarrow -\frac{n}{\beta} = 0$$

$$\Rightarrow \hat{\beta} = \infty \text{ (absurd result)}$$

If $x_{(1)}, x_{(2)} \ldots\ldots\ldots\ldots\ldots x_{(n)}$ is ordered sample then $-\beta \le x_{(1)} \le x_{(2)} \ldots\ldots \le x_{(n)} \le 0$.

We have to choose β so that L is maximum. Now L is

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maximum when β is minimum. Hence the minimum value of β consistent with the sample is. $x_{(1)}$, the smallest sample observation (i.e) $\hat{\beta} = x_{(1)}$.

Example : 2.1.16

Let $X_1, X_2, \ldots\ldots\ldots X_n$ denote a random sample from a bernouli distribution having the parameter p $(0 \le p \le 1)$. Find M.L.E. for p.

Solution :

The probability function for Bernouli distribution is

$$f(x,p) = nC_x p^x (1-p)^{n-x}, \qquad x = 0,1,\ldots n$$

Now, $$L = \prod_{i=1}^{n} \frac{n!}{x_i!(n-x_i)!} p^{x_i} (l-p)^{n-x_i}$$

$$= \frac{(n!)^n p^{\sum_i^n x_i} (1-p)^{n^2 - \sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i! \prod_{i=1}^n (n-x_i)!}$$

$$\log L = n \log n! + \sum_{i=1}^n x_i \log p + (n^2 - \sum_{i=1}^n x_i) \log(1-p) - \sum_{i=1}^n \log x_i! - \sum_{i=1}^n \log(n-x_i)!$$

Likelihood equation is

$$\frac{\partial}{\partial p} \log L = 0$$

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$$\Rightarrow \frac{1}{p}\sum_{i-1}^{n} x_i + (n^2 - \sum_{i-1}^{n} x_i)\frac{1}{(1-p)}(-1) = 0$$

$$\Rightarrow \frac{n\bar{x}}{p} - \frac{(n^2 - n\bar{x})}{1-p} = 0$$

$$\Rightarrow n\bar{x}(1-p) - p(n^2 - n\bar{x}) = 0$$

$$\Rightarrow n\bar{x} - np\bar{x} - pn^2 + pn\bar{x} = 0$$

$$\Rightarrow n\bar{x} - pn^2 = 0$$

$$pn^2 = n\bar{x}$$

$$p = \frac{\bar{x}}{n}$$

2.2 METHOD OF MOMENTS

The method of moments consists of equating the first few moments of a population to the corresponding moment of a sample, thus getting as many equations as are needed to solve for the unknown parameters of the population.

Definition :

The r^{th} sample moment of a set of observations x_1, x_2,……x_n is the mean of their powers and is denoted by

m_r^{I}

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(i.e) $m_r' = \frac{\sum_{i=1}^{n} x_i^r}{n}$

Therefore if a population has parameters the method of moments consists of solving the system of equation. $m_r^1 = \mu_r^1$, r = 1,2,…k for the k parameters.

Example 2.2.1

Let $x_1, x_2, \ldots\ldots x_n$ be a random of sample of size 'n' from a Poisson population with parameter λ, obtain the estimator of λ by the method of moments.

Solution :

The Poisson population has the probability function

$$p(x, \lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!}, x = 0,1,2,\ldots\ldots\ldots$$

$$\mu_1' = E(x) = \lambda$$

But $\bar{x} = \frac{x_1 + x_2 + \ldots\ldots\ldots x_n}{n}$

$$m_1' = \bar{x}$$

Solving the equation

$$m_1' = \mu_1' \Rightarrow \bar{x} = \lambda$$

Therefore the estimator of λ is $\bar{x}$

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Example 2.2.2

Obtain the estimator of the parameter p of the Binomial population using the method of moments.

Solution :

$P(x,n,p) = n_x p^x q^{n-x}$, $x = 0,1,2,\ldots n$

$$\mu_1' = E(x) = np$$

Let $x_1, x_2 \ldots . x_n$ be a random sample from the Binomial population then $m_1' = \frac{\sum x_i}{n} = \bar{x}$

Therefore by method of moments $m_1' = \mu_1'$

$$\bar{x} = np$$

$$p = \frac{\bar{x}}{n}$$

Therefore $\frac{\bar{x}}{n}$ is the estimator of p.

Example 2.2.3

Obtain the estimator of μ and σ^2 of the normal population using the method if moments.

Solution :

Let $x_1, x_2, \ldots . x_n$ be a random sample of size 'n' from the normal population.

N (μ , σ^2). The p.d.f of the Normal distribution.

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$$f(x, \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty < x < \infty$$

$$\mu_1^1 = E(x) = \mu$$

$$\sigma^2 = E(x^2) - [E(x)]^2$$

$$\sigma^2 = E(x^2) - \mu^2$$

$$m_2^1 = E(x^2) = \mu^2 + \sigma^2$$

But $m_1^1 = \frac{\sum x_i}{n} = \bar{x}$

$$m_2^1 = \frac{\sum x_i^2}{n}$$

Solving the equations $\mu_1^{'} = m_1^{'}$ and $\mu_1^{'} = m_2^{'}$

$$\mu_1^{'} = \bar{x}$$

$$\mu_2^{'} = m_2^{'}$$

$$\mu^2 + \sigma^2 = \frac{\sum x_i^2}{n}$$

$$\sigma^2 = \frac{\sum x_i^2}{n} - \mu^2$$

$$= \frac{1}{n}\sum x_i^2 - \bar{x}^2$$

$$\sigma^2 = \frac{1}{n}\sum (x_i - \bar{x})^2$$

The estimator of the parameter μ is $\bar{x}$

The estimator of the parameter σ^2 is $\frac{1}{n}\sum (x_i - \bar{x})^2$

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Example 2.2.4

The number of hours an electron tube will work is assumed to be an exponential variable with parameter λ. Given a sample of n life tubes of this sort, compute the method of moments estimator of λ.

Solution :

The pdf of the exponential distribution

$$f(x, \lambda) = \lambda e^{-\lambda x}, \quad x > 0,$$

$$0, \text{ otherwise}$$

$$\mu = \int_0^\infty x\lambda\ e^{-\lambda x}\, dx$$

$$= \lambda \int_0^\infty x\ e^{-\lambda x}\, dx$$

$$= \lambda \left[x\left(\frac{e^{-\lambda x}}{-\lambda}\right)_0^\infty - \int_0^\infty \frac{e^{-\lambda x}}{-\lambda} dx \right]$$

$$= \lambda \left[0 + \frac{1}{\lambda}\left(\frac{e^{-\lambda x}}{-\lambda}\right)_0^\infty \right]$$

$$= -\frac{1}{\lambda}(0-1)$$

$$= \frac{1}{\lambda}$$

Let x_1, x_2, ...x_n be a random sample from the population.

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Then $\bar{x} = \frac{x_1 + x_2 + \ldots x_n}{n}$, the method of moment estimator is given by $\bar{x} = \frac{1}{\lambda}$ (or) $\hat{\lambda} = \frac{1}{\bar{x}}$

Example 2.2.5

Find the estimator of θ of the uniform normal distribution having variable X in the interval $(-\theta, 0)$ by the method of moments.

Solution :

In uniform distribution first moment $\mu_1^{'} = E(x) = 0$. Hence we consider the second moment.

$$\mu_2^{'} = E(x^2) = \int_{-\theta}^{\theta} x^2 \left(\frac{1}{2\theta}\right) dx$$

$$= \frac{1}{2\theta}\left(\frac{x^3}{3}\right)_{-\theta}^{\theta}$$

$$= \frac{1}{6\theta}(\theta^3 + \theta^3) = \frac{2\theta^3}{6\theta} = \theta^2/3$$

Let $x_1, x_2 \ldots\ldots x_n$ be a random sample of size 'n' then

$$m_2^{'} = \frac{1}{n}\sum_{i=1}^{n} x_i^2$$

By the method of moments, solving the equation

$$m_2^{'} = \mu_2^{'}$$

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$$\frac{1}{n}\sum_{i=1}^{n} x_i^{\,2} = \frac{\theta^2}{3}$$

$$\theta^2 = 3 \quad \frac{\sum_{i=1}^{n} x_i^{\,2}}{n}$$

$$\theta^2 = 3\mu_2'$$

$$\theta = \sqrt{3\mu_2'}$$

$\therefore$ estimator of θ is $\sqrt{3\mu_2^1}$

Example 2.2.6

Let x_1, x_2.....x_n be a random sample from the distribution with p.d.f

$f(x,\theta) = \theta\, x^{\theta-1}$, $0 < x < 1$, $0 < \theta < \infty$. Find the estimator of θ by the method of moments.

$$\mu_1' = \int_0^1 x f(x,\theta)\,dx$$

$$= \int_0^1 x\theta x^{\theta-1}\,dx$$

$$= \theta\left[\left[x\left(\frac{x^\theta}{\theta}\right)\right]_0^1 - \int_0^1 \frac{x^\theta}{\theta}\,dx\right]$$

$$= \theta\left[\frac{1}{\theta} - \frac{1}{\theta}\left(\frac{x^{\theta+1}}{\theta+1}\right)_0^1\right]$$

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$$= \theta\left[\frac{1}{\theta} - \frac{1}{\theta}\left(\frac{1}{\theta+1}\right)\right]$$

$$= 1 - \frac{1}{\theta+1}$$

$$\mu_1' = \frac{\theta}{\theta+1}$$

$x_1, x_2, \ldots x_n$ be a random sample of size n.

Set $\quad m_1' = \frac{\sum_{i-1}^{n} x_i}{n}$

Set $\quad m_1' = \bar{x}$

Therefore by method of moments, solving the equation

$$\mu_1' = m_1'$$

$$\frac{\theta}{\theta+1} = \bar{x}$$

$$\frac{\theta+1}{\theta} = \frac{1}{\bar{x}}$$

$$1 + \frac{1}{\theta} = \frac{1}{\bar{x}}$$

$$\frac{1}{\theta} = \frac{1}{\bar{x}} - 1$$

$$\frac{1}{\theta} = \frac{1-\bar{x}}{\bar{x}}$$

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$$\theta = \frac{\bar{x}}{1-\bar{x}}$$

Hence the estimator of θ is $\frac{\bar{x}}{1-\bar{x}}$

Example 2.2.7

Estimate α and β in the case of Pearson's type III distribution by the method of moments.

$$f(x, \alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \ 0 \le x < \infty$$

$$\mu_r' = \int_o^{\infty} x^r . f(x, \alpha, \beta) dx$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_o^{\infty} x^r . \ x^{\alpha-1} \ e^{-\beta x} \ dx$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)} . \frac{\Gamma(\alpha + r)}{\beta^{d+r}}$$

$$\mu_r' = \frac{\beta^{\alpha}}{\Gamma(\alpha)} . \frac{\Gamma(\alpha + r)}{\beta^{d} \beta^{r}}$$

$$\mu_1' = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha).\beta} = \frac{\alpha}{\beta} \qquad \text{.......} (*)$$

Also $\mu_2' = \frac{\Gamma(\alpha + 2)}{\Gamma(\alpha).\beta^2}$

$$= \frac{(\alpha + 1)\alpha\Gamma(\alpha)}{\Gamma(\alpha)\beta^2}$$

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$$\mu_2' = \frac{(\alpha+1)\alpha}{\beta^2}$$

$$\frac{\mu_2'}{\mu_1'^2} = \frac{(\alpha+1)\alpha(\beta^2)}{\beta^2\alpha^2}$$

$$= \frac{\alpha+1}{\alpha}$$

$$= 1 + \frac{1}{\alpha}$$

$$\frac{1}{\alpha} = \frac{\mu_2' - \mu_1'^2}{\mu_1'^2}$$

$$\alpha = \frac{\mu_1'^2}{\mu_2' - \mu_1'^2}$$

Using (*) $\beta = \frac{\alpha}{\mu_1'}$

$$\beta = \frac{\mu_1'}{\mu_2' - \mu_1'^2}$$

Let x_1, x_2,….x_n be a random sample, whose moments are m_1' and m_2'

Therefore $\alpha = \frac{m_1'^2}{m_2' - m_1'^2}$

$$\beta = \frac{m_1'}{m_2' - m_1'^2}$$

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Example 2.2.8

For the double poission distribution p(x) = p(X=x)

$$= \frac{1}{2} \cdot \frac{e^{-m_1} m_1^x}{x!} + \frac{1}{2} \frac{e^{-m_2} m_2^x}{x!};$$

X = 0,1,2...

Show that the estimates for m_1 and m_2 by the method of moments are

$$\mu_1^1 \pm \sqrt{\mu_2{}^1 - \mu_1{}^1 - \mu_1{}^{1^2}}$$

Solution :

We have

$$\mu_1' = \sum_{x=0}^{\infty} xp(x) = \frac{1}{2} \cdot \frac{e^{-m_1} m_1^x}{x!} + \frac{1}{2} \sum_{x=0}^{\infty} \frac{e^{-m_2} m_2^x}{x!};$$

$$\mu_1' = \frac{1}{2} m_1 + \frac{1}{2} m_2 \quad \rightarrow \quad (*)$$

(Since the first second summations are the means of Poisson distributions with parameters m_1 and m_2 respectively).

$$\mu_2' = \sum_{x=0}^{\infty} x^2 p(x)$$

$$= \sum_{x=0}^{\infty} \frac{1}{2} \left[\sum_{x=0}^{\infty} x^2 \left(\frac{e^{-m_1} m_1^x}{x!} \right) + \sum_{x=0}^{\infty} x^2 \left(\frac{e^{-m_2} m_2^x}{x!} \right) \right]$$

$$= \frac{1}{2} \left[(m_1^2 + m_1) + (m_2^2 + m_2) \right]$$

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$$=\frac{1}{2}\left[(m_1+m_2)+(m_1^{2}+m_2^{2})\right] \qquad \text{.........} (*)$$

$$=\frac{1}{2}\left[2\mu_1^{'}+m_1^{2}+m_2^{2}\right] \qquad \text{from } (*)$$

$$=\frac{1}{2}\left[2\mu_1'+m_1^{2}+(2\mu_1^{'}-m_1)^2\right]$$

$$=\frac{1}{2}\left[2\mu_1^{'}+m_1^{2}+4\mu_1^{'2}+m_1^2-4\mu_1^{'}m_1\right]$$

$$=\mu_1'+\frac{1}{2}m_1^2+2\mu_1^{'2}+\frac{1}{2}m_1^2-2\mu_1^{'}m_1$$

$$\mu_2^{'}=\mu_1^{'}+m_1^2+2\mu_1^{'2}-2\mu_1^{'}m_1$$

$$m_1^2-2m_1\mu_1^{'}+2\mu_1^{'2}+\mu_1^{'}-\mu_2^{'})=0$$

$$\hat{m}_1=\frac{2\mu_1^{'}\pm\sqrt{4\mu_1^{'2}-4(2\mu_1^{'2}+\mu_1'-\mu_2')}}{2(1)}$$

$$=\mu_1^{'}\pm\sqrt{\mu_1^{'2}-2\mu_1^{'2}-\mu_1^{'}+\mu_2^{'}}\ s$$

$$\hat{m}_1=\mu_1'\pm\sqrt{(\mu_2^{'}-\mu_1'-\mu_1^{'2})}$$

Similarly substitute $m_1=2\mu_1^{'}-m_2$ in(*)

$$\mu_2^{'}=\frac{1}{2}\left[2\mu_1'+m_2^{2}+2\mu_1^{'}-m_2)^2\right]$$

$$=\frac{1}{2}\left[\ 2\mu_1^{'}+m_2^2+4\mu_1^{'2}-4\mu_1^{'}m_2+m_2^2\ \right]$$

$$\mu_2^{'}=\mu_1'+m_2^{2}+2\mu_1^{'2}-2\mu_1^{'}m_2$$

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$\Rightarrow m_2^2 - 2\mu_1' m_2 + (\mu_1' + 2\mu_1'^2 - \mu_2') = 0$

$$\hat{m}_2 = \frac{2\mu_1' \pm \sqrt{4\mu_1'^2 - 4(\mu_1' + 2\mu_1'^2 - \mu_2')}}{2(1)}$$

$$= \mu_1' \pm \sqrt{\mu_1'^2 - \mu_1' - 2\mu_1'^2 + \mu_2'}$$

$$\hat{m}_2 = \mu_1' \pm \sqrt{\mu_2' - \mu_1'^2 - \mu_1'}$$

Example 2.2.9

Obtain the estimator for θ in the case of a population with density function f(x) = (1 + θ) x $^{\theta}$ 0 ? x ? 1, θ > 0 by the method of moments.

Solution

Let $x_1, x_2, \ldots x_n$ be a random sample from the distribution with p.d.g f(x) = (1 + θ) x $^{\theta}$ 0 ? x ? 1.

$$\mu_1' = \int_0^1 x f(x,\theta) dx$$

$$= \int_0^1 x(1+\theta) x^{\theta} dx$$

$$= (1+\theta) \left(\frac{x^{(\theta+2)}}{(\theta+2)} \right)_0^1$$

$$= \left(\frac{(\theta+1)}{(\theta+2)} \right) (1-0)$$

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$$\mu_1' = \left(\frac{(\theta+1)}{(\theta+2)}\right)$$

Given $x_1, x_2, \ldots x_n$ be a random sample of size n.

Set $m_1' = \dfrac{\sum_{i-1}^{n} x_i}{n}$ i.e. $m_1' = \bar{x}$

$$\frac{(\theta+1)}{(\theta+2)} = \bar{x}$$

$$\frac{(\theta+2)-1}{(\theta+2)} = \bar{x}$$

$$1 - \frac{1}{\theta+2} = \bar{x}$$

$$1 - \bar{x} = \frac{1}{\theta+2}$$

$$\theta + 2 = \frac{1}{1-\bar{x}}$$

$$\theta = \frac{1}{1-\bar{x}} - 2$$

$$\theta = \frac{2\bar{x}-1}{1-\bar{x}}$$

Hence estimator of θ is $\dfrac{2\bar{x}-1}{1-\bar{x}}$

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2.3 METHOD OF LEAST SQUARES:

Suppose a sample of n observations (x_i, y_i) $i=1,2,3.....n$ is taken from a bivariate population with $(n+1)$ unknown parameters $a_0\ a_1.....a_n$ then, the method of least squares is used to fit a curve from $y= f(x,a_o, a_1 ....a_n)$
The sample observations will minimize the sum of squires of a residuals (i.e)

$$E = \sum_{i=1}^{n}[y_i - f(x,a_0,a_1.......a_n)]^2$$

subject to variation in $a_o, a_1 ....a_n$. The unknown parameters $a_o, a_1 ....a_n$ are estimated by using the following normal equations. $\frac{\partial E}{\partial a_i} = 0, i = 1,2,3,.....n$

Uses :

The method of least squares is used to fit

(i) linear regression

(ii) multiple linear regression

(iii) polynomial regression

(iv) exponential family of curves reducible to linear regression

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2.4 METHOD OF MINIMUM VARIANCE

Minimum variance unbiased estimator (M.V.U.E)

In this section we shall look for the estimators witch (i) are unbiased and (ii) have minimum variance

Let us take a sample of size n observations $x_1, x_2 \ldots\ldots x_n$ from a population with probability function $f(x,\theta)$ and the likelihood function be $L = \prod_{i=1}^{n} f(x_i, \theta)$

The problem is to find a statistic $t = t(x_1, x_2 \ldots x_n)$ such that

$$E(t) = \int_{-\infty}^{\infty} t.L dx = \gamma(\theta)$$

$$\Rightarrow \int_{-\infty}^{\infty} [t - \gamma(\theta)] L dx = 0 \quad \ldots\ldots(1)$$

$$\text{and var}(t) = \int_{-\infty}^{\infty} [t - E(t)]^2 L dx$$

$$= \int_{-\infty}^{\infty} [t - \gamma(\theta)]^2 L dx \quad \ldots\ldots\ldots\ldots(2)$$

is minimum, where

$\int_{-\infty}^{\infty} dx$ represents the n- fold integration

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots\ldots \int_{-\infty}^{\infty} dx_1 \, dx_2 \ldots dx_n$$

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(i.e) We have to minimize (2) subject to the condition

Remark :

This method is discussed in detail in creamer- Rao in equally

2.5 METHOD OF MINIMUM CHI – SQUARE ESTIMATION

In this method we choose the estimators for the population parameters in such a way to minimize

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} \qquad \text{.......(1)}$$

where E_i's are expected frequencies calculated from this population.

Likelihood function, for getting O_1 members in the first cell and O_2 members in the second cell ...... and O_r in the r^{th} cell, is obtained by using the multinomial distribution

$$L = \frac{n!}{O_1!O_2!......O_r!}\left(\frac{E_1}{O_1}\right)^{O_1}\left(\frac{E_2}{O2}\right)^{O_2}.......\left(\frac{E_r}{O_r}\right)^{O_r}$$

$$\therefore \log L = \text{Constant} + \sum_i O_i \log\left(\frac{E_i}{O_i}\right)$$

For large samples, we have $E_i = O_i + A_i n^{1/2}$ where A_i is finite, also ,$A_{i\,i} < O_i$

$| A_i n^{1/2} | < O_i$ and $\sum A_i = 0$

$$\therefore \log L = C + \sum_{i=1}^{r} O_i \log\left[\frac{O_i + A_i\, n^{1/2}}{O_i}\right]$$

$$= C + \sum_{i=1}^{r} O_i \log\left[1 + \frac{A_i\, n^{1/2}}{O_i}\right]$$

$$= C - \frac{1}{2} \sum O_i \left[\frac{A_i^2\, n}{O_i^2} + o(n^{-1/2})\right]$$

$$= C - \frac{1}{2} \sum \frac{A_i^2\, n}{O_i} + o(n^{-1/2})$$

$$\log L = C - \frac{1}{2}\chi'^2 + O(n^{-1/2})$$

modified version of (1) is $\chi'^2 = \sum \frac{(O_i - E_i)^2}{O_i}$ ....(2)

$$\log L = C - \frac{1}{2} \sum \frac{(O_i - E_i)^2}{O_i} + o(n^{-1/2})$$

$$\chi^2 - \chi'^2 = \sum (O_i - E_i)^2 \left(\frac{1}{E_i} - \frac{1}{O_i}\right)$$

$$= \sum (O_i - E_i)^2 \left[(O_i + A_i\, n^{1/2})^{-1} - \frac{1}{O_i}\right]$$

$$= \sum \frac{(O_i - E_i)^2}{O_i} \left[O_i (O_i + A_i\, n^{1/2})^{-1} - 1\right]$$

$$= \sum \frac{(O_i - E_i)^2}{O_i} \left[(1 + A_i\, n^{1/2}/O_i)^{-1} - 1\right]$$

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$= o(n^{-1/2})$ ....(3)

From (2)and (3) for large samples up to order $n^{-1/2}$ the estimators obtained by minimizing χ'^2 or χ'^2 are equivalent to the estimators obtained by maximizing the likelihood sum.

INTERVAL ESTIMATION

Interval estimation for large samples

As per central limit theorem, for large samples, the underlying distribution of standardized variate corresponding to the sampling distribution of the sample statistic 't' will be asymptotically normal (ie) $z = \frac{t - E(t)}{S.E(t)} \sim$ N(0,1) is standardized variate with mean zero ad variance 1 as $n \to \infty$.

2.6 CONFIDENCE INTERVAL OF THE MEAN

Let μ be the population mean and $\bar{x}$ be the sample mean of the sampling distributions of means. Assume that the sample mean is normal (or) the sample is large. Then the interval estimate of population mean by the sample mean $\bar{x}$ of the sampling distribution of means is given by the following rule.

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Working Rule :

Step 1 : Calculate $\bar{x}$ (or) take $\bar{x}$.

Step 2 : Select the confidence level

Step 3 : compute S.E $(\bar{x})$ with the help of following results.

Case (i) :

when σ,the S.D of the normal population is known

S.E.$(\bar{x}) = \frac{\sigma}{\sqrt{n}}$, where n is sample size and n is large.

Case (ii) :

When σ is not known S.E.$(\bar{x}) = \frac{s}{\sqrt{n-1}}$,

s = standard deviation of the sampling distribution of samples.

Step 4:

Construct a confident level as follows :

Case I :

When σ is known and population is normal or any population with large 'n'.

confidence interval = $\{\bar{x}$ - S. E$(\bar{x})$ x Z_α, $\bar{x}$ + S.E $(\bar{x})$ $Z_\alpha\}$

$$\text{(or) } [\bar{x} - \frac{\sigma}{\sqrt{n}} \times Z_\alpha \leq \mu \leq \bar{x} + \frac{\sigma}{\sqrt{n}} \times Z_\alpha]$$

When σ is nonknown and n is large. In this case the confidence interval is

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$$\left[\bar{x} - \frac{s}{\sqrt{n-1}} \times Z_\alpha \le \mu \le \bar{x} + \frac{s}{\sqrt{n-1}} \times Z_\alpha\right]$$

Step 5. Select the value of Z_α from the following table

Confidence level (l-α)	Value of confidence co-efficient Z_α (or) t_α
90%	1.64
95%	1.96
98%	2.33
99%	2.58
Without any reference to the confidence level	3.00

Example 2.6.1

The quality control manager of tyre company has sample of 100 tyres and has found the mean life time to be 30,214 km. The population S.D. is 860. Construct a 95% confidence interval for the mean life time for this particular brand of tyres.

Solution :

Here n = 100 $\bar{x}$=30,214 σ = 860 Also (l-α) % = 95% so that Z_α = 1.96

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$$\text{S.E.}(\bar{x}) = \frac{\sigma}{\sqrt{n}} = \frac{860}{\sqrt{100}} = 86$$

$$\therefore \text{ confidence interval} = [\bar{x} - \frac{\sigma}{\sqrt{n}} \times Z_{\alpha},\ \bar{x} + \frac{\sigma}{\sqrt{n}} \times Z_{\alpha}]$$

$$= [30214\text{-}86 \times 1.96, 30214+86 \times 1.96]$$

$$= [30045.44,\ 30382.56]$$

Example 2.6.2

In a random selection of 64 of 600 road crossing in a town the mean number of automobile accidents per year was found to be 4.2 and the sample S.D was 0.8 construct a 95% confident interval for the mean number of automobile accidents per crossing per year.

Solution :

Here n = 64 Sample mean $\bar{x} = 42$

Sample S.D s = 0.8 N = 600 and poplation size is finite.

$\frac{n}{N} = \frac{64}{600} = 0.106 > 0.05$ so we use the following result to calculate S.E of $\bar{x}$.

$$\text{S.E.}(\bar{x}) = \frac{s}{\sqrt{n-1}} \times \sqrt{\frac{N-n}{N-1}} = \frac{0.8}{\sqrt{63}} \times \sqrt{\frac{600-64}{600-1}}$$

Also (l-α) % = 95% $\therefore Z_{\alpha} = 1.96 = 0.091s$

Confidence interval = $[\bar{x} - \text{S.E.}(\bar{x}) \times Z_{\alpha},\ \bar{x} + \text{S.E.}(\bar{x}) \times Z_{\alpha}]$

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$= [4.2 - 0.091 \times 1.96, 4.2 + 0.091 \times 1.96]$

$= [4.023, 4.377]$

Example 2.6.3

If a sample is 20, population standard deviation is 3 and sample size is 64, find the internal of the mean.

Solution :

Here n = 64, $\bar{x}$ = 20 σ =3

Interval estimate of the sample mean is $\bar{x} \pm \frac{\sigma}{\sqrt{n}} \times 1.96$

$$20 \pm \frac{3}{\sqrt{64}} \times 1.96$$

Hence the interval estimate of the mean is [19.265, 20.735]

Example 2.6.4

A sample of 100 gave a mean of 7.4 kg and a S.D. of 1.2 kg. Find 95% confidence limits for population mean

Solution

Here n = 100 $\bar{x}$ = 7.4 kg s = $\hat{\sigma}$ = 1.2

$$S.E(\bar{x}) = \frac{\hat{\sigma}}{\sqrt{n}} = \frac{1.2}{\sqrt{100}} = 0.1$$

95% confidence limits for population mean are given by

$$\bar{x} \pm 1.96 \frac{\hat{\sigma}}{\sqrt{n}}, = 7.4 \pm \frac{1.2}{\sqrt{100}} \times 1.96$$

∴ confidence limits are 7.1648 and 7.6352

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Remark:

Finite population correction factor $\frac{n}{N}$ is used in finding confidence limits, if $\frac{n}{N} > 0.05$ and the sample size 'n' is large.

2.7 CONFIDENCE INTERVAL ESTIMATE OF THE PROPORTION

Definition :

Population proportion P is the ratio of the number of elements possessing a characteristic to the total number of elements in the population (i.e)

$$P = \frac{\text{Number of elements possessing the characteristic}}{\text{Total number elements in the sample}}$$

Definition :

Sample proportion p is the ratio of the number of the elements possessing a characteristic to the total number of elements n in the sample.

Important Note :

The mean of sampling distribution of p equals the population proportion (ie) E(p) = p

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Standard error of sample preparation

Population size	Sample drawn	Standard error of sample proportion (p)
Infinitely large	With replacement	S.E. (p) = $\sqrt{\frac{PQ}{n}}$, if P is known where $Q = 1 - P$ S.E ($\hat{p}$)= $\sqrt{\frac{pq}{n}}$ if P is not known where q = 1-p, n, samples size. It is an estimated value
Finite population	Without replacement	S.E. (p) = $\sqrt{\frac{PQ}{n}}$ $\sqrt{\frac{N-n}{N-1}}$, if P is known Q= 1-P, N = size of the population. S.E. (p)= $\sqrt{\frac{pq}{n}} \times \sqrt{\frac{N-n}{N-1}}$, if P is not known It is an estimated rate.

Confident interval estimate

Confidence interval estimates of the population proportion P. is given by the following results.

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Case i :

When P is known. In this case the confidence interval of sample proportion is given by [p - Z_α × S.E. (p), p + Z_α × S.E. (p)]

Where Z_α is the confidence co-efficient and S.E (p) is the standard error of p.

Case ii.

When P is not known. In this case the confidence interval of the sample proportion is given by [p-S.E(p) × Z_α, p + S.E. (p) × Z_α] where Z_α= confidence co-efficient; standard error of p = S.E. (p)

Example 2.7.1

A random sample of 800 units from a large consignment, showed that 200 were damaged. Find 95% confidence limits for the population proportion of damaged units in the consignment.

Solution :

Here n = 100 sample proportion $p = \frac{200}{800} = 0.25$

$$q = 1 - p = 1 - 0.25 = 0.75$$

$$\text{S.E. (p)} \sqrt{\frac{qp}{n}} = \sqrt{\frac{0.25 \times 0.75}{100}} = 0.031$$

Also (1-α) % = 95% so that $Z_\alpha = 1.96$

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Confidence limits for the population

Proportion are $p \pm Z_{\alpha} \times S.E.(p)$

$$= 0.25 \pm 1.96 \times 0.031$$

Lower limit $= 00.25 - 1.96 \times 0.031$

$$= 0.19$$

Upper Limit $= 0.25 + 1.96 \times 0.031$

$$= 0.31$$

Hence 95% confidence limits for the population proportion of damaged units in the consignment are 0.19 and 0.31.

Example 2.7.2

A random sample of 500 pineapples was taken from the large consignment and 65 of them were found to be bad. Show that is the S.D of the population of badone in a sample of this size is 0.015 and deduce the % of bad pineapples in the consignment lie between 8.5 and 17.5

Solution :

Here n = 500, sample proportion $p = \frac{65}{500} = 0.13$

$q = 1 - p = 0.87$

S.E. of population of bad pineapples

$$S.E(p) = \sqrt{\frac{pq}{n}} = \sqrt{\frac{0.13 \times 0.87}{500}} = 0.015$$

Here the level is not specific so we take z = 3. Therefore 99.73% confidence limits for the proportion of bas pineapples

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$= p \pm S.E\ (p)\ Z_\alpha$

$= 0.13 \pm 3 \times 0.0015$

$= 0.13\ \pm 0.045$ (ie.) 0.085 to 0.175

Hence 99.73% confidence limits for bad pineapples lies between 0.085×100 and 0.175×100 (i.e) between 8.5% and 17.5%

Example 2.7.3

In a health survey involving a random sample of 90 patients who developed a particular illness, 80% of them are cured of this illness by a new drug. Establish 90% confidence interval for the population proportion of all the patients, who will be cured by the new drug. This would help to assess the market potential for this new drug by the pharmaceutical company.

Solution :

Here n = 90 Sample proportion p = 0.80

q = 1-p=1-0.80 = 0.20, $(1-\alpha)\% = 90\% = 1.64$

$$S.E\ (p) = \sqrt{\frac{pq}{n}} = \sqrt{\frac{0.8 \times 0.2}{90}} = 0.042$$

90% confidence interval population proportion cured by the new drug is

$p \pm S.E.\ (p)\ Z_\alpha$(i.e) $0.8\ \pm 0.042 \times 1.64$

$0.8 \pm\ 0.0688$ (I,e) 0.7312 to 0.8688

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Exercise Problems

1. The mean height obtained from a sample of size 100 taken randomly from a population is 164 cms. If the standard deviation of the height distribution of the population is 3 cm. Set up 95% and 99% confidence limits for the mean height of the population,

   (Hint here n = 100, $\bar{x}$ = 164, σ =3, z = 1.96

   $$\text{S.E.} = \frac{\sigma}{\sqrt{n}} = 0.3$$

   Confidence limits 164 ± 1.96 (0.3) = 164 ± 0.588

   (ie) 163.412 to 164. 588

2. A random sample of size 65 was taken to estimate the mean annual income of 1000 families and the mean and S.D. were found to be Rs. 6300 and Rs. 9.5 respectively. Find (a). 95% (b). 98% confidents intervals for the population.

3. Out of 300 households In a town 123 have TV. Sets. Find 95% confidence limits to the true value of the proportion of the households with T.V Sets in the whole town.

(Hint : n = 300, sample proportion $p = \frac{123}{300} = 0.41$ S.E

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(p) $\sqrt{\frac{pq}{n}}$ =0.0283, confidence limits. $0.41 \pm 1.96 \times 0.0283$, (0.355, 0.465)

2.8 CONFIDENCE INTERVALS

Definition :

Suppose the p.d.f of a distribution depends on unknown parameter θ. Two statistics L and U functions of a random sample from the distribution is defined to form of 100 (1 -α)% confidence interval for θ if $P[L \leq \theta \leq U] = 1 - \alpha$, $0 \leq \theta \leq 1$ 1- α is said to be confidence co –efficient.

The following results may be useful during calculating confidence intervals.

R_1 : If X is N (μ,σ^2) then $\frac{\bar{x}-\mu}{\sigma}$ ~N(0,1)

R_2 : If X is N (μ,σ^2) then X^2 is χ^2 (1)

R_3 : If X_1 and X_2 are independent random variables and X_1 is $N(\mu_1, \sigma_1^2)$ and X_2 is $N(\mu_2, \sigma_2^2)$ than $X_1 \pm X_2$ is $N(\mu_1 \pm \mu_2, \sigma_1^2 + \sigma_2^2)$

R_4 : If X_1 and X_2 are independent with X_1 is $\chi^2(n_1)$ and X_2 is $\chi^2(n_2)$ then

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(a) $X_1 + X_2$ is χ^2 (n_1/n_2)

(b) $\dfrac{X_1/n_1}{X_1/n_2}$ is F (n_1, n_2)

R_5: If X and Y are independent random variables with X is N(0,1) and Y is χ^2 (r), then

$X/\sqrt{y/r}$ is student's density with r. d.f (i.e.) t(r).

R_6 : If X is random variable that is b(n,p) $0 \le p \le 1$,

then $Z = \dfrac{X/n - p}{\sqrt{\frac{X}{n}(1-\frac{X}{n})/n}}$ is N (0,1) for large n.

When a random sample $x_1, x_2 \ldots . x_n$ is drawn from $N(\mu, \sigma^2)$ we have the following important results.

S_1 : $\overline{X}$ is $N(\mu, \sigma^2/n)$

S_2 : $\dfrac{\sum_{i=1}^{n}(x_i - \mu)^2}{\sigma^2}$ is $\chi^2(n)$

(Since for i = 1,2,....n is N (μ, σ^2) we get

$(x_i - \mu)/\sigma$ is N(0,1) (by R_1).

Applying R_2, $\left(\dfrac{x_i - \mu}{\sigma}\right)^2$ is $\chi^2(1)$. Applying R_4 we get

$\dfrac{1}{\sigma^2}\sum(x_i - \mu)^2$ is $\chi^2(n)$.

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S_3 : $\dfrac{ns^2}{\sigma^2}=\dfrac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x})^2$ is $\chi^2(n-1)$

S_4 : $\dfrac{\bar{x}-\mu}{s/\sqrt{n-1}}$ is t (n-1)

(Q from S_1, we have $\bar{x}$ is N $(\mu,\sigma^2/n)$) Using R_1,

we get $\dfrac{\bar{x}-\mu}{\sigma/\sqrt{n}}$ is N(0,1) . Using this and S_3 in R_5

we get $\dfrac{\bar{x}-\mu}{\sigma/\sqrt{n}}\Big/\sqrt{\dfrac{ns^2}{\sigma^2(n-1)}}$ is t (n-1).

In finding confidence intervals for unknown parameters, we will be repeatedly using this above results.

Example 2.8.1

Obtain 100(1-α)% confidence interval for the unknown parameter p of a binomial distribution, when the parameter n is known.

Solution :

Let the discrete random variable X follows binomial distribution with known parameter'n'a +ve integer,o< p< 1.

From R_6, $z=\dfrac{\left(\dfrac{X}{n}\right)-p}{\sqrt{\left(\dfrac{X}{n}\right)(1-X/n)/n}}$ is N (0,1)

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From Normal area table, we can find a number $Z_{1-\alpha/2}$ for given α, such that

$$P\left[-Z_{1-\frac{\alpha}{2}} \le z \le z_{1-\alpha/2}\right] = 1-\alpha.$$

$$P\left[-Z_{1-\frac{\alpha}{2}} \le \frac{\left(\frac{X}{n}\right)-p}{\sqrt{\left(\frac{X}{n}\right)(1-X/n)/n}} \le z_{1-\frac{\alpha}{2}}\right] = 1-\alpha$$

$$P\left[-Z_{1-\frac{\alpha}{2}}\sqrt{\frac{X}{n}(1-X/n)/n} \le \left(\frac{X}{n}\right)-p \le Z_{1-\frac{\alpha}{2}}\sqrt{\frac{X}{n}(1-X/n)/n}\right]$$

$$P\left[-\frac{X}{n}-Z_{1-\frac{\alpha}{2}}\sqrt{\frac{X}{n}(1-X/n)/n} \le -p \le -\frac{X}{n}+Z_{1-\frac{\alpha}{2}}\sqrt{\frac{X}{n}(1-X/n)/n}\right]$$

$$= 1 - \alpha$$

$$P\left[\frac{X}{n}+Z_{1-\frac{\alpha}{2}}\sqrt{\frac{X}{n}(1-X/n)/n} \ge p \ge \frac{X}{n}-Z_{1-\frac{\alpha}{2}}\sqrt{\frac{X}{n}(1-X/n)/n}\right]$$

$$= 1 - \alpha$$

$$P\left[\frac{X}{n}-Z_{1-\frac{\alpha}{2}}\sqrt{\frac{X}{n}(1-X/n)/n} \le p \le \frac{X}{n}+Z_{1-\frac{\alpha}{2}}\sqrt{\frac{X}{n}(1-X/n)/n}\right]$$

$$= 1 - \alpha$$

Therefore confidence interval is

$$\left[\frac{X}{n}-Z_{1-\frac{\alpha}{2}}\sqrt{\frac{X}{n}\left(1-\frac{X}{n}\right)/n}, \frac{X}{n}+Z_{1-\frac{\alpha}{2}}\sqrt{\frac{X}{n}\left(1-\frac{X}{n}\right)/n}\right]$$

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Example 2.8.2

A survey on service stations and garages showed that out of 412 tire pressure gauges tested 239 gave accurate readings. We are interested in finding 99% confidence interval for the proportion of all such tire gauges that give accurate readings.

Solution :

Let X denote the number of tyre gauges that give accurate readings. Here X = 239, n = 412, 1-α = 99. From, normal area tables $Z_{0.01} = 2.58$.

Therefore the confidence interval

$$\left[\frac{239}{412} - 2.58\sqrt{\frac{239}{412}(1-\frac{239}{412})/412},\right.$$

$$\left.\frac{239}{412} + 2.58\sqrt{\frac{239}{412}(1-\frac{239}{412})/412}\right]$$

$$\left[0.5801 - 2.58\sqrt{0.5801(1-0.5801)/412}\right.$$

$$\left.0.5801+2.58\sqrt{0.5801(1-0.5801)/412}\right]$$

[0.5801-0.0627,0.5801+0.027]

(0.5174, 0.6428)

Obtain 100 (1-α)% confidence interval for the difference $\mu_1 - \mu_2$ between the means of two independent normal distributions say $N(\mu_1,\sigma^2)$ and $N(\mu_2,\sigma^2)$ when the distributions have the same but unknown variance σ^2.

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Solution :

Let $x_1, x_2 \ldots\ldots x_n$ be a random sample from $N(\mu_1, \sigma^2)$

If X is $N(\mu_1, \sigma^2)$ then $\overline{X}$ is $N(\mu_1, \sigma^2/n)$ (by S_1)

Let $y_1, y_2 \ldots\ldots y_m$ be a random sample from $N(\mu_2, \sigma^2)$

If Y is $N(\mu_2, \sigma^2)$ then $\overline{Y}$ is $N(\mu_2, \sigma^2/m)$ (by S_1)

$\overline{X}$ is $N(\mu_1, \sigma^2/n)$ and $\overline{Y}$ is $N(\mu_2, \sigma^2/m)$

$$\Rightarrow \overline{X} - \overline{Y} \text{ is } N\left(\mu_1 - \mu_2 , \frac{\sigma^2}{n} + \frac{\sigma^2}{m}\right) \text{ (by } R_3)$$

$$\Rightarrow (\overline{X} - \overline{Y} - (\mu_1 - \mu_2)) / \sqrt{\frac{\sigma^2}{n} + \frac{\sigma^2}{m}} \, is N(0,1) \text{ [by } R_1] \rightarrow (1)$$

ns_1^2 / σ^2 is $\chi^2(n-1)$ [By S_3] → (2)

ms_2^2 / σ^2 is $\chi^2(m-1)$ [By S_3] → (3)

$$\Rightarrow \text{ where } S_1^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2$$

$$S_2^2 = \frac{1}{m}\sum_{i=1}^{m}(Y_i - \overline{Y})^2$$

From (2) and (3)

$\Rightarrow (nS_1^2 + mS_2^2)/\sigma^2$ is $\chi^2(n-1+m-1)$ [By R_4 (a)]

$$\text{Let } X = ((\overline{X} - \overline{Y}) - (\mu_1 - \mu_2)) / \sqrt{\frac{\sigma^2}{n} + \frac{\sigma^2}{m}} \rightarrow (4) \text{ and}$$

$Y = (nS_1^2 + mS_2^2) / \sigma^2$

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Thus we get

$$T = \frac{(\overline{X}-\overline{Y}-(\mu_1-\mu_2))/\sigma\sqrt{\frac{1}{n}+\frac{1}{m}}}{\sqrt{\frac{nS_1^2+mS_2^2}{\sigma^2}/(n+m-2)}} \text{ is } t(n+m-2)$$

.... by R_5

$$T = \frac{(\overline{X}-\overline{Y}-(\mu_1-\mu_2))}{\sqrt{\left(\frac{1}{n}+\frac{1}{m}\right)}\sqrt{(nS_1^2+mS_2^2)/(n+m-2)}} \text{ is } t(n+m-2)$$

$$T = \frac{(\overline{X}-\overline{Y}-(\mu_1-\mu_2))}{\sqrt{\left(\frac{1}{n}+\frac{1}{m}\right)}\sqrt{(nS_1^2+mS_2^2)/(n+m-2)}} \text{ is } t(n+m-2)$$

From the table for students 't' distribution we can find.

$t_{1-\alpha/2}\ (n+m-2)$ for given α such that

$P(-t_{1-\alpha/2}\ (n+m-2) \le T \le t_{1-\alpha/2}(n+m-2)] = 1-\alpha \quad \rightarrow (5)$

Let $D = \sqrt{\frac{ns_1^2+ms_2^2}{n+m-2}\left(\frac{1}{n}+\frac{1}{m}\right)}$ we have

$$T = \frac{(\overline{X}-\overline{Y})-(\mu_1-\mu_2)}{D}$$

From (5) we have

$$P\left[-t_{1-\frac{\alpha}{2}}(n+m-2) \le \frac{(\overline{X}-\overline{Y})-(\mu_1-\mu_2)}{D} \le t_{1-\frac{\alpha}{2}}(n+m-2)\right] = 1-\alpha$$

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$$P\left[-(\overline{X}-\overline{Y})-t_{1-\frac{\alpha}{2}}(n+m-2).D \le -(\mu_{1}-\mu_2) \le -(\overline{X}-\overline{Y})+t_{1-\frac{\alpha}{2}}(n+m-2).D\right]$$

$$= 1-\alpha$$

$$P\left[(\overline{X}-\overline{Y})+t_{1-\frac{\alpha}{2}}(n+m-2).D \ge (\mu_1-\mu_2) \ge (\overline{X}-\overline{Y})-t_{1-\frac{\alpha}{2}}(n+m-2).D\right]$$

$$= 1-\alpha$$

$$P\left[(\overline{X}-\overline{Y})-t_{1-\frac{\alpha}{2}}(n+m-2).D \le (\mu_1-\mu_2) \le (\overline{X}-\overline{Y})+t_{1-\frac{\alpha}{2}}(n+m-2).D\right]$$

$$= 1-\alpha$$

Therefore required confidence interval is

$$\left[(\overline{X}-\overline{Y})-t_{1-\frac{\alpha}{2}}(n+m-2).D,\ (\overline{X}-\overline{Y})+t_{1-\frac{\alpha}{2}}(n+m-2).D\right]$$

Example 2.8.3

The miles per gallon achieved by car A, were 20.7, 24.0, 23.2, 21.6, 21.0 and similar results for car B are 25.8, 23.7, 23.4

Solution :

These data yield

$$n = 5,\ \sum_{i=1}^{5} x_i = 110.5,\ \sum_{i=1}^{5} x_i^2 = 2450.29$$

$$m = 3,\ \sum_{i=1}^{3} y_i = 72.9,\ \sum_{i=1}^{3} y_i^{\,2} = 1774.89$$

$$\text{So, } \bar{x} = \frac{110.5}{5} = 22.1 \quad \sum_{i=1}^{5}(x_i-\bar{x})^2 = 8.24$$

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$$\bar{y} = \frac{72.9}{3} = 24.3 \quad \sum_{i=1}^{3}(y_i - \bar{y})^2 = 3.42$$

$$S_1^2 = \frac{1}{n}\sum_{i=1}^{5}(x_i - \bar{x})^2 \quad S_2^2 = \frac{1}{m}\sum_{i=1}^{3}(y_i - \bar{y})^2$$

$$= \frac{8.24}{5} = 1.648 \quad = 3.24/3 \ - 1.14$$

$$D = \sqrt{\frac{nS_1^2 + ms_2^2}{(n+m-2)}\left(\frac{1}{n} + \frac{1}{m}\right)}$$

$$= \sqrt{\frac{5x1.648 + 3x1.14}{(5+3-2)}\left(\frac{1}{5} + \frac{1}{3}\right)}$$

The required 90% confidence interval at (n+m-2) = 6 d.f is given by

(22.1-24.3 – 1.943 x 1.0180, 22.1-24.3+1.943x1.018)

= (-4.1779, -0.22)

Obtain 100 (l-α)% confidence interval for the ratio of variances of σ_1^2 / σ_2^2 based on two independent samples from normal populations. N(μ_1, σ_1^2) and N(μ_2, σ_2^2)

Solution :

Consider a random sample $X_1, X_2 \ldots\ldots\ldots X_n$ of size n $\geq$2.From N (μ_1, σ_1^2) and random sample. Y_1, $Y_2 \ldots\ldots\ldots Y_n$ of size. m$\geq$ 2. from N(μ_2,σ_2^2).

Let $\bar{x}$ and $\bar{y}$ be the respective sample means. Let S_1^2 and S_2^2 the sample variances.

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Using result S_3, nS_1^2/σ_1^2 and $\frac{mS_2^2}{\sigma_2^2}$ have

independent chi-square distribution with (n-1) and (m-1) d.f respectively.

Using R_4 (b), we have

$$U = \frac{nS_1^2/\sigma_1^2(n-1)}{mS_2^2/\sigma_2^2(m-1)}$$ is distributed as F(n-1, m-1).

From the table of F distribution, we can find two constants.

$$F_{\alpha/2}(n-1, m-1) \text{ and } F_{1-\alpha/2}(n-1, m-1)$$

Such that

$$P[F_{\alpha/2}(n-1, m-1) < U < F_{1-\alpha/2}(n-1, m-1)] = 1-\alpha.$$

$$P[F_{\alpha/2}(n-1, m-1) < \frac{nS_1^2\sigma_2^2(m-1)}{\sigma_1^2(n-1)ms_2^2} < F_{1-\alpha/2}(n-1, m-1)] = 1-\alpha.$$

$$\Rightarrow P\left[\frac{mS_2^2(n-1)}{nS_1^2(m-1)}F_{\alpha/2}(n-1,m-1) < \frac{\sigma_2^2}{\sigma_1^2} < F_{1-\alpha/2}(n-1,m-1)\frac{mS_2^2(n-1)}{nS_1^2(m-1)}\right]$$

But $F_{\alpha/2}(n-1,m-1) = \dfrac{1}{F_{1-\alpha/2}(m-1,n-1)}$

Therefore $100(1-\alpha)\%$ confidence interval for $\frac{\sigma_2^2}{\sigma_1^2}$ is

$$\left[\frac{mS_2^2(n-1)}{nS_1^2(m-1)}\left(\frac{1}{F_{1-\alpha/2}(m-1,n-1)}\right), \frac{mS_2^2(n-1)}{nS_1^2(m-1)}F_{1-\alpha/2}(n-1,m-1)\right]$$

Example 2.8.4

A sample of size n = 13, items drawn from N (μ_1, σ_1^2) has

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mean $\bar{x} = 18.97$, and $\sum_{1}^{13}(x_i - \bar{x})^2 = 128.41$. Another sample of size m = 9, drawn from N (μ_2, σ_2^2) has mean $\bar{y} = 23.20$, and $\sum_{1}^{9}(y_i - \bar{y})^2 = 36.72$. Find 99% confidence interval for the ratio σ_1^2/σ_2^2.

Solution :

100 (1-α)% confidence interval for $\frac{\sigma_2^2}{\sigma_1^2}$ is

$$\left[\frac{mS_2^2(n-1)}{nS_1^2(m-1)}\cdot\frac{1}{F_{1-\alpha/2}(m-1,n-1)},\frac{mS_2^2(n-1)}{nS_1^2(m-1)}.F_{1-\alpha/2}(n-1,m-1)\right]$$

100(1-α)% confidence interval for $\frac{\sigma_1^2}{\sigma_2^2}$ is

$$\left[\frac{nS_1^2(m-1)}{mS_2^2(n-1)}\cdot\frac{1}{F_{1-\alpha/2}(n-1,m-1)},\frac{nS_1^2(m-1)}{mS_2^2(n-1)}.F_{1-\alpha/2}(m-1,n-1)\right]$$

$$n = 13, \quad S_1^2 = \frac{\sum_{i=1}^{13}(x_i - \bar{x})^2}{13}$$

$$13\,S_1^2 = \sum_{i=1}^{13}(x_i - \bar{x})^2 = 128.41,$$

$$m = 9,\ S_2^2 = \frac{\sum_{j=1}^{9}(y_j - \bar{y})^2}{9} \Rightarrow 9\,S_2^2 = \sum_{j=1}^{9}(y_i - \bar{y})^2 = 36.72$$

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Substitute in (1)

$$\left[\frac{128.41(8)}{36.72(12)}\cdot\frac{1}{5.67},\frac{(128.41)8}{(36.72)(12)}.4.5\right]$$

$$\left[\frac{1027.28}{2498.42},\frac{4622.76}{440.64}\right]$$

[0.4112, 10.491]

Obtain (1-α)% confidence interval for the parameters (a) θ (b) o² of the normal distribution.

Solution :

Let x_i (i = 1, 2,…n) be a random sample of size 'n' from the density f(x : θ, σ) and let

$$\bar{x} = \frac{1}{n}\sum_{i-1}^{n} x_i,\ s^2 = \frac{1}{n}\sum_{i-1}^{n}(x_i - \bar{x})^2$$

$$S^2 = \frac{1}{n-1}\sum_{i-1}^{n}(x_i - \bar{x})^2$$

(a). The statistic $t = \dfrac{\bar{x}-\theta}{S/\sqrt{n}}$

follows students 't' distribution with (n-1) degrees of freedom. Hence 100 (1 - α)% confidence limits for θ are given by $P[|\,t\,| \le t_\alpha] = 1-\alpha$

$$\Rightarrow P\left[\left|\frac{\bar{x}-\theta}{S\sqrt{n}}\right| \le t_\alpha\right] = 1-\alpha$$

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$$\Rightarrow P\left[|\bar{x}-\theta| \leq \frac{S}{\sqrt{n}} t_\alpha\right] = 1 - \alpha\sigma$$

$$\Rightarrow P\left[\bar{x} - t_\alpha \frac{S}{\sqrt{n}} \leq \theta \leq \bar{x} + t_\alpha \frac{S}{\sqrt{n}}\right] = 1 - \alpha$$

where t_α is the tabulated value of t for (n-1) degrees of freedom at significance level α. Hence the required confidence interval for θ is :

$$(\bar{x} - t_\alpha \frac{S}{\sqrt{n}}, \bar{x} + t_\alpha \frac{S}{\sqrt{n}})$$

(b). Case (i) θ is known, and assume $\theta = \mu$

Then $\frac{\sum (x_i - \mu)^2}{\sigma^2} = \frac{ns^2}{\sigma^2} \sim \chi^2$ (n)

Define χ^2_α as the value of χ^2 such that

$P(\chi^2 > \chi^2_\alpha) = \int_{\chi^2_\alpha}^{\infty} P(\chi^2) d\chi^2 = \alpha$ .... (1) where

$P(\chi^2)$ is the p.d.f of χ^2 distribution with n d.f then the required confidence interval is given

$$P\left[\chi^2_{1-(\alpha/2)} \leq \chi^2 \leq \chi^2_{(\alpha/2)}\right] = 1 - \alpha$$

$$\Rightarrow P\left[\chi^2_{1-(\alpha/2)} \leq \frac{ns^2}{\sigma^2} \leq \chi^2_{(\alpha/2)}\right] = 1 - \alpha \quad \text{.............. (2)}$$

Now $\frac{ns^2}{\sigma^2} \leq \chi^2_{(\alpha/2)}$

$$\Rightarrow \frac{ns^2}{\chi^2_{\alpha/2}} \leq \sigma^2$$

and $\chi^2_{1-(\alpha/2)} \leq \frac{ns^2}{\sigma^2} \Rightarrow \sigma^2 \leq \frac{ns^2}{\chi^2_{1-(\alpha/2)}}$

from (2) $P[ns^2/\chi^2_{(\alpha/2)} \leq \sigma^2 \leq \frac{ns^2}{\chi^2_{1-(\alpha/2)}}] = 1 - \alpha$ ...(3)

where $\chi^2_{1-(\alpha/2)}$ and $\chi^2_{(\alpha/2)}$ are obtained from (1) by using n d.f.

For example 95% confidence interval for σ^2 is given by

$$P\left[\frac{ns^2}{x^2_{0.025}} \leq \sigma^2 \leq \frac{ns^2}{x^2_{0.975}}\right] = 0.95$$

Case (ii) 'θ' is unknown. In this case the stsatistic

$$\frac{\sum(x_i - \bar{x})^2}{\sigma^2} = \frac{ns^2}{\sigma^2} \sim \chi^2_{(n-1)}$$

Here also confidence interval for σ^2 is given by (3)

where χ^2_{α} is the significant value of χ^2 (as defined in (1)) for (n-1) degrees of freedom at the significance level α.

Example 2.8.5

Given one observation from a population with p.d.f

$f(x,\theta) = \frac{2}{\theta^2}(\theta - x), 0 \leq x \leq \theta$. Obtain 100 (1-α)%.

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confidence intervals for θ.

Solution :

The density of $u = x/\theta$ is given by

$$g(u) = f(x, \theta) \left|\frac{dx}{du}\right| \text{ is given by}$$

$$= \frac{2}{\theta^2}(\theta - x).\ \theta$$

$$= \frac{2}{\theta}(\theta - x)$$

$$= 2(1 - \frac{x}{\theta})$$

$$= 2(1 - u), \quad 0 \le u \le 1$$

To obtain 100 (1-α)% confidence interval for θ, we choose two quantities u_1 and u_2 such that

$P[u_1 \le u \le u_2] = 1 - \alpha$ ....(1)

and $P[u < u_1]$ and $p[u > u_2] = \alpha / 2$

$$P[u \le u_1] = \frac{\alpha}{2}$$

$$\Rightarrow \int_0^{u_1} 2(1-u)\, du = \frac{\alpha}{2}$$

$$2\left(u - \frac{u^2}{2}\right)_o^{u_1} = \frac{\alpha}{2}$$

$$2\left(u_1 - \frac{u_1^2}{2}\right) = \frac{\alpha}{2}$$

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$$2u_1 - u_1^2 = \frac{\alpha}{2}$$

$$4u_1 - 2u_1^2 = \alpha$$

$$2u_1^2 - 4u_1 + \alpha = 0$$

$$u_1^2 - 2u_1 + \frac{\alpha}{2} = 0 \ldots\ldots\ldots\ldots (*)$$

Similarly

$$P[u > u_2] = \frac{\alpha}{2}$$

$$\Rightarrow \int_{u_2}^{1} 2(1-u)du = \frac{\alpha}{2}$$

$$2\left[u - \frac{u^2}{2}\right]_{u_2}^{1} = \frac{\alpha}{2}$$

$$2\left[(1 - \frac{1}{2}) - (u_2 - \frac{u_2^2}{2})\right] = \frac{\alpha}{2}$$

$$1 - 2(u_2 - \frac{u_2^2}{2}) = \frac{\alpha}{2}$$

$$2 - 4u_2 + 2u_2^2 = \alpha$$

$$u_2^2 - 2u_2 + (1 - \frac{\alpha}{2}) = 0 \ldots\ldots\ldots\ldots (**)$$

From (1)

$$P[u_1 \le \frac{x}{\theta} \le u_2] = 1 - \alpha$$

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$$P[\frac{1}{u_1} \geq \frac{\theta}{x} \geq \frac{1}{u_2}] = 1 - \alpha$$

$$P[\frac{x}{u_1} \geq \theta \geq \frac{x}{u_2}] = 1 - \alpha$$

$$P[\frac{x}{u_2} \leq \theta \leq \frac{x}{u_1}] = 1 - \alpha$$

Hence required interval for θ is $\left(\frac{x}{u_1}, \frac{x}{u_2}\right)$

Where u_1 and u_2 are given by * and **

Example 2.8.6

Show that the largest observations L of a sample n observations from a rectangular distribution with density function $f(x, \theta) = \frac{1}{\theta}, 0 \leq x \leq \theta$

$= 0$, Otherwise

has the distribution

$$dG(L) = n\left(\frac{L}{\theta}\right)^{n-1} \bullet \frac{dL}{\theta}, \quad 0 \leq L \leq \theta$$

Show that the distribution of $V = L/\theta$ is given by p.d.f

$h(v) = nv^{n-1}, 0 \leq v \leq 1$

Hence deduce that the confidence limits for θ corresponding to confidence. co-efficient α are

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L and $\dfrac{L}{(1-\alpha)^{1/n}}$

Solution : Let $X_1, X_2 \ldots\ldots\ldots\ldots X_n$ be a random sample of size 'n' from the population and let L = max $(X_1, X_2, \ldots\ldots\ldots\ldots X_n)$. The distribution of L is given by $dG(L) = n[F(L)]^{n-1}.\, f(L)\, dL$ where F(.) is the distribution function of X given by

$$F(L) = \int_0^L f(x,\theta)dx$$

$$= \int_0^L \frac{1}{\theta}dx = \frac{L}{\theta}$$

Therefore $dG(L) = n\left[\dfrac{L}{\theta}\right]^{n-1} .\dfrac{1}{\theta}.dL, \quad 0 \le L \le \theta$

Take $V = \dfrac{L}{\theta}$, the jacobian of the transformation is θ.

Hence p.d.f h(.) of V s given by

$$h(v) = nv^{n-1}.\,\frac{1}{\theta}.\theta.\ 0 \le v \le 1$$

$$= nv^{n-1}, \quad 0 \le v \le 1$$

To obtain confident limits for θ, with confidence co-efficient α, let us define v_α such that

$P(v_\alpha < V < 1) = \alpha$ .... (1)

$$\Rightarrow \int_{v_\alpha}^{1} h(v)dv = \alpha$$

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$$\int_{V_\alpha}^{1} n v^{n-1} dv = \alpha$$

$$n\left(\frac{v^n}{n}\right)_{v_\alpha}^{1} = \alpha$$

$$1 - v_\alpha^n = \alpha$$

$$v_\alpha^n = 1-\alpha$$

$$v_\alpha = (1-\alpha)^{1/n} \quad \ldots. \qquad (2)$$

Sub (2) in (1)

$$P[(1-\alpha)^{1/n} < V < 1) = \alpha$$

$$\Rightarrow \quad P[(1-\alpha)^{1/n} < \frac{L}{\theta} < 1] = \alpha$$

$$\Rightarrow \quad P[\frac{1}{(1-\alpha)^{1/n}} > \frac{\theta}{L} > 1] = \alpha$$

$$\Rightarrow \quad P[\frac{L}{(1-\alpha)^{1/n}} > \theta > L] = \alpha$$

$$\Rightarrow \quad P[L < \theta < \frac{L}{(1-\alpha)^{1/n}}] = \alpha$$

Hence the required confidence limits for θ are L and L / $(1-\alpha)^{1/n}$.

Example 2.8.7

Give a random sample from a population with p.d.f $f(x,\theta) = \frac{1}{\theta}$, $0 \leq x \leq \theta$ show that 100 $(1-\alpha)$% confidence

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intervals for θ is given by R and R/ψ is given by $\psi^{n-1}[n-(n-1)\psi] = \alpha$ and R is the sample range.

Solution :

The joint p.d.f of $x_1, x_2 \ldots\ldots\ldots x_n$ is given by

$$L = \frac{1}{\theta^n}, 0 \le x_i \le \theta$$

If $x_{(1)}, x_{(2)} \ldots\ldots x_{(n)}$ is the ordered sample then the joint p.d.f of $x_{(n)}$ and $x_{(1)}$ is given by

$$g(x_{(1)}, x_{(n)}) = \frac{n(n-1)}{\theta^n}[x_{(n)} - x_{(1)}]^{n-2},\ 0 \le x_{(1)} \le x_{(n)} \le \theta$$

To obtain the distribution of the sample range R. Let us make the transformation of the variables

$R = x_{(n)} - x_{(1)}$ and $v = x_{(1)}$

Therefore $v = x_{(n)} - R \le \theta - R$

Jacobian of the transformation

$$\frac{\partial(R,v)}{\partial(x_{(n)},x_{(1)})} = \begin{vmatrix} \frac{\partial R}{\partial x_{(n)}} & \frac{\partial R}{\partial x_{(1)}} \\ \frac{\partial v}{\partial x_{(n)}} & \frac{\partial v}{\partial x_{(1)}} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = 1$$

The joint p.d.f of R and V becomes

$$h(R,V) = \frac{n(n-1)}{\theta^{n-1}}.R^{n-2}, 0 < v \le \theta - R$$

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The marginal density of R is given by

$$h_1(R) = \int_0^{\theta-R} h(R,v)dv$$

$$= \int_0^{\theta-R} \frac{n(n-1)}{\theta^n} R^{n-2} dv$$

$$= \frac{n(n-1)}{\theta^n} R^{n-2}(\theta - R), 0 \le R \le \theta$$

The density of U = R/θ is

$$h_2(u) = h_1(R)\left|\frac{dR}{du}\right|$$

$$= \frac{n(n-1)}{\theta^n} R^{n-2}(\theta - R)\theta$$

$$= \frac{n(n-1)}{\theta^{n-1}}(\theta R^{n-2} - R^{n-1})$$

$$= n(n-1)\left[\left(\frac{R}{\theta}\right)^{n-2} - \left(\frac{R}{\theta}\right)^{n-1}\right]$$

$$= n(n-1)[u^{n-2} - u^{n-1}]$$

$$= n(n-1)u^{n-2}(1-u), 0 \le u \le 1$$

100 (l-α)% confidence interval for θ is given by

$$P(\psi \le U \le 1) = l - \alpha \quad ....(1)$$

$$\text{Now} \int_0^{\psi} h_2(u)du = \alpha$$

$$\Rightarrow n(n-1)\int_0^{\psi} u^{(n-2)}(1-u)du = \alpha$$

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$$n(n-1)\left(\frac{u^{n-1}}{n-1}-\frac{u^{n}}{n}\right)_{0}^{\psi}=\alpha$$

$$\Rightarrow n(n-1)\left(\frac{\psi^{n-1}}{n-1}-\frac{\psi^{n}}{n}\right)=\alpha$$

$$\Rightarrow \psi^{n-1}[n-(n-1)\psi]=\alpha \qquad \dots \qquad (2)$$

From (1)

$$P[\psi \le \frac{R}{\theta} \le 1]=1-\alpha$$

$$P[\frac{1}{\psi} \ge \frac{\theta}{R} \ge 1]=1-\alpha$$

$$\Rightarrow P[1 \le \frac{\theta}{R} \le \frac{1}{\psi}]=1-\alpha$$

$$P[R \le \theta \le \frac{R}{\psi}]=1-\alpha$$

Hence the required limits for θ are given by R and R/ ψ, where ψ is the solution of (2).

Obtain 100 (1-α)% confidence intervals for population mean μ, based on a random sample of size n from N(μ,σ^2) when (i) σ^2 is known (ii) σ^2 unknown.

Solution :

(i). Since we have to find confidence interval for μ, consider MLE of μ which is sample mean $\bar{x}$. We

know that if $\bar{x}$ is $N(\mu, \frac{\sigma^2}{n})$. Also we know that $\bar{x}$ is $N(\mu, \sigma^2)$ then $Z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$ is N (0,1). We can find a number $Z_{1-(\alpha/2)}$ for a given α, such that

$$P[-Z_{1-\alpha/2} \le \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \le Z_{1-\alpha/2}] = 1-\alpha$$

For example if $\alpha = 0.05$, then

$Z_{1-\alpha/2} = Z_{0.975} = 1.96$ and if $\alpha = 0.10$

then $Z_{1-\alpha/2} = Z_{0.9} = 1.645$

We have the following in equation

$$-Z_{1-\alpha/2} \le \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \le Z_{1-\alpha/2}$$

$$-Z_{1-\alpha/2} \le (\frac{\sigma}{\sqrt{n}}) \le \bar{x} - \mu \le Z_{1-\alpha/2}(\sigma/\sqrt{n})$$

$$-\bar{x} - Z_{1-\alpha/2}(\sigma/\sqrt{n}) \le -\mu \le Z_{1-\alpha/2}(\sigma/\sqrt{n}) - \bar{x}$$

$$\bar{x} + Z_{1-\alpha/2}(\sigma/\sqrt{n}) \ge \mu \ge \bar{x} - Z_{1-\alpha/2}(\sigma/\sqrt{n})$$

$$P[\bar{x} - Z_{1-\alpha/2}(\sigma/\sqrt{n}) \le \mu \le \bar{x} + Z_{1-\alpha/2}(\sigma/\sqrt{n})] = 1-\alpha$$

Therefore 100(1-α)% confidence interval for population mean μ is

$$[\bar{x} - Z_{1-\alpha 2}(\sigma/\sqrt{n}), \bar{x} + Z_{1-\alpha 2}(\sigma/\sqrt{n})]$$

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Example 2.8.8

Let X be the life time of an electric bulb manufactured by certain manufacturer of light bulbs in N(μ, 1296). If a random sample of 27 bulbs were tested until they burned out, yielding a sample man of $\bar{x}$=1478 hours find 95% confidence interval for μ.

$$\left[\bar{x} - Z_{1-\alpha 2}(\sigma/\sqrt{n}), \bar{x} + Z_{1-\alpha 2}(\sigma/\sqrt{n})\right]$$

$$\left[1478 - 1.96x\frac{36}{\sqrt{27}}, 1478 + 1.96x\frac{36}{\sqrt{27}}\right]$$

=[1464.42, 1491.58]

(ii) When σ^2 is unknown,

We know that $\frac{\bar{x} - \mu}{S/\sqrt{n-1}}$ follows t distribution with (n-1) d.f where

$$S^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

Using the table for students 't' distribution given in the annexure we can find $t_{1-(\alpha/2)}$ (n-1) d.f for a give α, such that

$$P[-t_{1-\alpha/2}(n-1) \leq \frac{\bar{x} - \mu}{S/\sqrt{n-1}} \leq t_{1-\alpha/2}(n-1)] = 1 - \alpha$$

The following inequalities are equivalent,

$$[-t_{1-\alpha/2}(n-1) \leq \frac{\bar{x} - \mu}{S/\sqrt{n-1}} \leq t_{1-\alpha/2}(n-1)$$

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$$[-t_{1-\alpha/2}(n-1)\frac{S}{\sqrt{n-1}} \le \bar{x}-\mu \le t_{1-\alpha/2}(n-1)\frac{S}{\sqrt{n-1}}$$

$$-\bar{x}-t_{1-\alpha/2}(n-1)\frac{S}{\sqrt{n-1}} \le -\mu \le -\bar{x}+t_{1-\alpha/2}(n-1)\frac{S}{\sqrt{n-1}}$$

$$\bar{x}+t_{1-\alpha/2}(n-1)\frac{S}{\sqrt{n-1}} \ge \mu \ge \bar{x}-t_{1-\alpha/2}(n-1)\frac{S}{\sqrt{n-1}}$$

$$\bar{x}-t_{1-\alpha/2}(n-1)\frac{S}{\sqrt{n-1}} \le \mu \le \bar{x}+t_{1-\alpha/2}(n-1)\frac{S}{\sqrt{n-1}}$$

Therefore

$$P\left[\bar{x}-t_{1-\alpha/2}(n-1)\frac{S}{\sqrt{n-1}} \le \mu \le \bar{x}+t_{1-\alpha/2}(n-1)\frac{S}{\sqrt{n-1}}\right] = 1$$

and 100 (1-α) % confidence interval for μ is

$$\bar{x}-t_{1-\alpha/2}(n-1)\frac{S}{\sqrt{n-1}},\ \bar{x}+t_{1-\alpha/2}(n-1)\frac{S}{\sqrt{n-1}}]$$

Example 2.8.9

The weights of 10 people of a locality are found to be 70, 67, 62, 68, 61, 68, 70, 64, 64, 66 Kilogram. Find 95% confidence level.

Solution :

$$\text{Here } n = 10,\ \bar{x} = \frac{660}{10} = 66$$

X :	70	67	62	68	61	68	70	64	64	66
$x-\bar{x}$:	4	1	-4	2	-5	2	4	-2	-2	0
$(x-\bar{x})^2$:	16	1	16	4	25	4	16	4	4	0

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$$S^2 = \frac{1}{n}\sum(x-\bar{x})^2$$

$$= \frac{90}{10} = 9$$

$S = 3$, $t_{0.05}$ at (10-1= 9 d.f) = 2.262

Therefore 95% confidence limits for population mean

$$[\bar{x} - t_{0.05}(9)\frac{S}{\sqrt{n-1}}, \bar{x} + t_{0.05}(9)\frac{S}{\sqrt{n-1}}]$$

$$[66 - 2.262 \times \frac{3}{\sqrt{9}}, 66 + 2.262 \times \frac{3}{\sqrt{9}}]$$

[63.738, 68.262]

2.9 CONFIDENCE INTERVAL FOR POPULATION MEAN FOR SMALL SAMPLES USING 'T' DISTRIBUTION

If $t_{0.05}$ is the tabulated value of t for v = n-1 d.f at 5% level of significance (ie) $P(|t| > t_{0.05}) = 0.05$

$\Rightarrow P(|t| \leq t_{0.05}) = 0.95$

The 95% confidence limits for μ are given by $|t| \leq t_{0.05}) = 0.95$ (ie) $\left|\frac{\bar{x}-\mu}{S\sqrt{n}}\right| \leq t_{0.05}$

$$\Rightarrow \bar{x} - t_{0.05}\frac{S}{\sqrt{n}} \leq \mu \leq \bar{x} + t_{0.05}\frac{S}{\sqrt{n}}$$

Thus 95% confidence limits for μ are $\bar{x} \pm t_{0.05}\frac{S}{\sqrt{n}}$

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Similarly 99% confidence limits for μ are $\bar{x} \pm t_{0.01} \frac{S}{\sqrt{n}}$

Where $t_{0.01}$ is the tabulated value of t for $v = (n-1)$ d.f at 1% level of significance.

Example 2.9.1

A random sample of 10 boys had the following I.Q's 70,120,110,101,88,83,95,98,107,100. Find a reasonable range in which most of the mean I.Q values of samples of 10 boys lie.

Solution :

x	$x-\bar{x}$	$(x-\bar{x})^2$
70	-27.2	739.84
120	22.8	519.84
110	12.8	163.84
101	3.8	14.44
88	-9.2	84.64
83	-14.2	201.64
95	-2.2	4.84
98	0.8	0.64
107	9.8	96.04
100	2.8	7.84
972		1833.60

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Here n = 10, $\bar{x} = \frac{972}{14} = 97.2$ and

$$S^2 = \frac{1}{(n-1)}\sum(x-\bar{x})^2$$

$$= \frac{1833.60}{9} = 203.73$$

95% confidence limits within which the mean I.Q values of sample of 10 boys will lie are given by

$$\bar{x} \pm t_{0.05}\ \frac{S}{\sqrt{n}}$$

$$= 97.2 \pm 2.262 \times 4.514$$

$$= 97.2 \pm 10.21$$

$$= 107.41 \text{ and } 86.99$$

Hence the required 95% confidence interval is (86.99, 107.41)

Example 2.9.2

The average travel time taken based on a random sample of 10 people working in a company to reach the office is 40 minutes with a S.D of 10 minutes. Establish the 95% confidence interval for the mean travel time of everyone in the company and re-design the working hours.

Solution :

The 1-α confidence interval for the population mean is given by

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$$\bar{x} - t_{n-1}\frac{S}{\sqrt{n}} \leq \mu \leq \bar{x} + t_{n-1}\frac{S}{\sqrt{n}}$$

Here n = 10, $\bar{x}$ = 40 s = 10 for 95% confidence interval

$$\frac{\alpha}{2} = \frac{0.05}{2} = 0.025$$

Each tail will have probability of 0.025 is 2.262

$$(40 - 2.262\left(\frac{10}{\sqrt{10}}\right) \leq \mu \leq 40 + 2.62\ \sqrt{10})$$

$$(40 - 7.1530 \leq \mu \leq 40 + 7.1530)$$

$$(32.847 \leq \mu \leq 47.1530)$$

Example 2.9.3

A soap manufacturing company was distributing a particular band of soap through a large number of retail shops. Before a heavy advertisement campaign, the mean sales per week per shop were 140 dozens. After the campaign, a sample of 26 shops was taken and the mean sale was found to be 147 dozens with SD =16. Can you ascertain the confidence limits after the campaign for this soap manufacturing company.

Solution :

Here μ = 140, n = 26 $\bar{x}$ = 147. s = 16 v = n-1 = 25

95% confidence limits for population mean are

$$\bar{x} \pm t_{n-1}\frac{S}{\sqrt{n}}$$

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$$147 \pm 2.06 \times \frac{16}{\sqrt{26}}$$

$$147 \pm 6.464$$

140.536 and 153.464

The value of t at 5% of level of significance for n-1=25 d.f from the appendix table 2, 2.06.

Example 2.9.4

A server channel monitered for an hour was found to have an estimated mean of 20 transactions transmitted per minute. The variance is known to be 4. Find the standard error.Establish an interval estimate that include a population mean 95% of time and 99% of time.

Solution :

(a) population variance $\sigma^2 = 4 \Rightarrow \sigma = 2$

no of samples n = 1 hour = 60 min

sample mean $\bar{x}$ = 20 transaction / min

standard error $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = 0.2582$

confidence level = 95%

1- α = 95%

From the tables $z_{\alpha} = 1.96$

Confidence limits for population mean is

$$\bar{x} \pm z_{\alpha} \frac{\sigma}{\sqrt{n}} = (20 \pm 1.96 \times 0.2582) = (20 \pm 0.5060)$$

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(ie) 19.494 and 20.506

Approximating μ we get 19.5 to 20.5 transaction per minute, 0.5 value of transaction means one way of transaction is complete as a transaction in two way.

Level of confidence = 99%

$1 - \alpha = 99\%$

$1 - \alpha = 99\% \Rightarrow z_\alpha = 2.58$ (from table)

$\mu = 20 \pm 2.58\ (0.2582)$

$= 20 \pm 0.6662$

$\mu = (19.338, 20.6662)$

2.10 CONFIDENCE INTERVALS FOR LARGE SAMPLES

The following result is helpful to find confidence interval for large sample

$$z = \frac{\frac{\partial}{\partial\theta}\log L}{\sqrt{\operatorname{var}(\frac{\partial}{\partial\theta}\log L)}} \sim N(0,1)$$

Thus for large samples, the confidence interval for θ with confidence co-efficient $(1-\alpha)$ is obtained by converting the inequalities in $p[|z| \leq \lambda_\alpha] = 1-\alpha$ where λ_α is given by

$$\frac{1}{\sqrt{2\pi}} \int_{-\lambda_\alpha}^{\lambda_\alpha} \exp(-u^2/2)du = 1-\alpha$$

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Example 2.10.1

Obtain 100 (1-α)% confidence limits (for large samples) for the parameter λ of the poission distribution.

$$f(x,\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \; x = 0,1,2..........$$

Solution : The likelihood function is given by

$$L = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$$

$$= e^{-n\lambda} \; \frac{\lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}$$

$$\log L = -n\lambda + \sum_{i=1}^{n} x_i \log \lambda - (\sum_{i=1}^{n} \log x_i!)$$

$$\frac{\partial}{\partial\lambda} \log L = -n + \frac{1}{\lambda} \sum_{i=1}^{n} x_i$$

$$= -n + \frac{n\bar{x}}{\lambda}$$

$$= n(\frac{\bar{x}}{n} - 1)$$

$$\text{Var}\,(\frac{\partial}{\partial\lambda}\log L) = E(-\frac{\partial^2}{\partial\lambda^2}\log L)$$

$$E(\frac{n\bar{x}}{\lambda^2}) = \frac{n}{\lambda^2} E(\bar{x})$$

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$$= \frac{n}{\lambda^2} . \lambda$$

$$= \frac{n}{\lambda}$$

$$z = \frac{\frac{\partial}{\partial \lambda} \log L}{\sqrt{E(-\frac{\partial^2}{\partial \lambda^2} \log L)}}$$

Therefore z $= \dfrac{n(\frac{\bar{x}}{\lambda} - 1)}{\sqrt{\frac{n}{\lambda}}}$

$$= \frac{n(\bar{x} - \lambda)}{\lambda\sqrt{\frac{n}{\lambda}}}$$

$$= \sqrt{\frac{n}{\lambda}}(\bar{x} - \lambda) \sim N(0,1)$$

Hence 100 (1-α) % confidence interval for λ is given by (for large sample)

$$P[|z| \leq \lambda_{\alpha}] = 1\text{-}\alpha$$

$$P[|\sqrt{\frac{n}{\lambda}}(\bar{x} - \lambda) \leq \lambda_{\alpha} 1] \quad 1\text{-}\alpha$$

Hence the required limits for λ are the roots of the equation

$$|\sqrt{\frac{n}{\lambda}}(\bar{x} - \lambda)| = \lambda_{\alpha}$$

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$$\frac{n(\bar{x}-\lambda)^2}{\lambda} = \lambda_\alpha^2$$

$$n(\bar{x}-\lambda)^2 = \lambda.\lambda_\alpha^2$$

$$\nu(\bar{x}^2+\lambda^2-2\lambda\bar{x}) = \lambda.\lambda_\alpha^2$$

$$\lambda^2-2\lambda\bar{x}+\bar{x}^2) = \frac{\lambda.\lambda_\alpha^2}{n}$$

$$\lambda^2-\lambda(2\bar{x}+\frac{\lambda_\alpha^2}{n})+\bar{x}^2 = 0$$

$$\lambda = \frac{(2\bar{x}+\frac{\lambda_\alpha^2}{n}) \pm \sqrt{(2\bar{x}+\frac{\lambda_\alpha^2}{n})-4\bar{x}^2}}{2} \text{s}$$

95% confidence interval for λ is given by taking $\lambda_\alpha = 1.96$

given by

$$\lambda = (\frac{2\bar{x}+3.84}{n} \pm \sqrt{\frac{4\bar{x}\lambda_\alpha^2}{n}+\frac{\lambda_\alpha^4}{n^2}})/2$$

$$= (\bar{x}+\frac{1.9208}{n}) \pm \sqrt{\frac{3.8146\bar{x}}{n}+\frac{3.689}{n^2}}$$

$$= \bar{x} \pm 1.96\sqrt{\frac{x}{n}} \text{ to order } n^{-1/2}.$$

Example 2.10.2

Show that for the distribution $dF(x) = \theta\, e^{-x\theta}$, $0 < x < \infty$ central confidence limits for θ for large sample with 95% confidence co-efficient are given by

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$$\theta = (1 \pm \frac{1.96}{\sqrt{n}}) / \bar{x}$$

Solution :

Let x_{1n} x_2 .... x_n be a random sample of size 'n'

$$L = \prod_{i=1}^{n} f(x_i, \theta)$$

$$= \prod_{i=1}^{n} \theta . e^{-x_i \theta}$$

$$L = \theta^n \quad e^{-\theta \sum_{i=1}^{n} x_i}$$

$$\log L = n \log \theta - \theta \sum_{i=1}^{n} x_i$$

$$\frac{\partial}{\partial \theta} \log L = \frac{n}{\theta} - \sum_{i=1}^{n} x_i$$

$$= n \left(\frac{1}{\theta} - \bar{x}\right)$$

$$\frac{\partial^2}{\partial \theta^2} \log L = n\left(-\frac{1}{\theta^2}\right) = -\frac{n}{\theta^2}$$

$$\text{Var} \left(\frac{\partial}{\partial \theta} \log L\right) = E\left(-\frac{\partial^2}{\partial \theta^2} \log L\right)$$

$$= E\left(+\frac{n}{\theta^2}\right) = \frac{n}{\theta^2}$$

Hence for large samples,

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$$Z=\frac{\frac{\partial}{\partial\theta}\log L}{\sqrt{\text{var}(\frac{\partial}{\partial\theta}\log L)}}=\frac{n(\frac{1}{\theta}-\bar{x})}{\sqrt{\frac{n}{\theta^2}}}\sim N(0,1)$$

$$=\sqrt{n}(1-\theta\bar{x})\sim N(0,1)$$

Hence 95% confidence limits for θ are given by

$P[|z|\leq\theta_\alpha]=1-\alpha$

$\Rightarrow P[-1.96\leq\sqrt{n}(1-\theta\bar{x})\leq 1.96]=0.95$

Now $\sqrt{n}(1-\theta\bar{x})\leq 1.96$

$\Rightarrow \sqrt{n}-(1.96)\leq\sqrt{n}\theta\bar{x}$

$\Rightarrow \frac{\sqrt{n}-1.96}{\sqrt{n}\bar{x}}\leq\theta$

$\Rightarrow \left(1-\frac{1.96}{\sqrt{n}}.\frac{1}{\bar{x}}\right)\leq\theta$ .... (1)

Also, $-1.96\leq\sqrt{n}.(1-\theta\bar{x})$

$-1.96\leq\sqrt{n}-\sqrt{n}\theta\bar{x})$

$\Rightarrow \sqrt{n}\theta\bar{x}\leq\sqrt{n}+1.96$

$\Rightarrow \theta\leq(1+\frac{1.96}{\sqrt{n}})\frac{1}{\bar{x}}$ ...(2)

Hence the required limits, λ for are the roots of the equation

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Example 2.10.2

Show that for the distribution DF(x) = o<x<∞ central confidence limits for q for large samples with 95% confidence co-efficient are given by

$\Rightarrow$ p-1.96 $\leq 1.96 \leq \sqrt{n}\theta\bar{x}$

$$\Rightarrow \frac{\sqrt{n}-1.96}{\sqrt{n}\bar{x}} \leq \theta$$

$$\Rightarrow (1-\frac{1.96}{\sqrt{n}}.\frac{1}{\bar{x}}) \leq \theta \qquad \rightarrow (1)$$

Also $-1.96 \leq \sqrt{n}\,(1-\theta\bar{x})$

$$\Rightarrow -1.96 \leq \sqrt{n}-\sqrt{n}\theta\bar{x}$$

$$\Rightarrow \sqrt{n}\theta\bar{x} \leq \sqrt{n}+1.96$$

$$\Rightarrow \theta \leq (1+\frac{1.96}{\sqrt{n}})\frac{1}{\bar{x}} \qquad \rightarrow (2)$$

2.11 EXERCISE

1. A random sample of size 16 from N(μ ,25) yielded $\overline{X}$ = 73.8.Find a 95% confidence interval for μ.

   Ans: (71.35,76.25)

2. Assume that the yield per acre for particular variety of soybeans is N(μ ,σ^2).For a random sample of 5 plots, the yields per acre were 37.5, 48.8, 46.9, 55.0,

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44.0.Find 90% confidence interval for μ. Ans: (40.36,52.587)

3. In 37 samples each containing 100 milliliter of water, number of bacterial calories were counted. The sample data yielded $\overline{X}$ = 11.95, S – 11.80 (in units of hundreds of calories).Find the approximate 95% confidence interval for mean number of calories Ans : (8.15,15.75)
4. A random sample of size from $N(\mu,\sigma^2)$ yielded $\bar{x}$ = 89.7 and σ^2 = 11.72, Fine a 95% confidence interval of σ^2. Answer (6.69, 25.637)
5. If 8.6, 7.9, 8.3, 6.4, 8.4, 9.8, 7.2, 7.8, 7.5 are the observed values of a random, sample of size 9 from a distribution that is N $(8,\sigma^2)$ construct a 90% confidence interval for σ^2. (Answer (0.43, 2.21)
6. Let Y be a (300,p)..If the observed value of Y is 75 find an approximate 95% confidence interval for p. (Answer : (0.001, 0.299)
7. The number of defectives in a sample of 112 chips is 59. Approximate 95% confidence interval for the fraction defective in the lot. Ans : (0.425, 0.619)
8. For two independent sample yielded n = 10, m = 7, $\bar{x}$ = 4.2, $\bar{y}$ = 3.4 , S_1^2 = 49, S_2^2 = 32. Find the

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confidence interval for $\mu_1 - \mu_2$. Answer (-5.16, 6.76)

9. For developing countries in Asia (excluding china) and Africa respectively, let p_1 and p_2 be the proportion of prescribed children with chronic malnutrition (stunting). If respective random samples of $n_1 = 1300$ and $n_2 = 1100$ yielded 520 and 385 children with chronic malnutrition. Find approximate 95% confidence interval for $p_1 - p_2$ (Answer [0.011, 0.0987)

10. Let X and Y equal to weights of a phosphorous free laundry detergent in two different type of makes respectively. Assume X and Y are $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$ respectively. A random sample of 10 observations of X yielded a sample variance 0.0-040 while a random sample of 9 observations of Y yielded a sample variance of 0.0076. Find a 95% confidence interval for σ_1^2/σ_2^2 Answer : (0.121, 2.158)

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